On the Helly Property of Some Intersection Graphs*

Tanilson D. Santos[1,2,3],
Advisors: Jayme L. Szwarcfiter[1,2], **Uéverton S. Souza**[4], **Claudson F. Bornstein**[2]

[1]Programa de Engenharia de Sistemas e Computação (PESC/COPPE - UFRJ)

[2]Universidade Federal do Rio de Janeiro (UFRJ)

[3]Universidade Federal do Tocantins (UFT)

[4]Universidade Federal Fluminense (UFF)

tanilson.dias@uft.edu.br, {jayme@nce, cfb@dcc}.ufrj.br, ueverton@ic.uff.br

Abstract. *An EPG graph G is an edge-intersection graph of paths on a grid. In this thesis, we analyze structural characterizations and complexity aspects regarding EPG graphs. Our main focus is on the class of B_1-EPG graphs whose intersection model satisfies well-known the Helly property, called Helly-B_1-EPG. We show that the problem of recognizing Helly-B_1-EPG graphs is NP-complete. Besides, other intersection graph classes such as VPG, EPT, and VPT were also studied. We completely solve the problem of determining the Helly and strong Helly numbers of B_k-EPG graphs and B_k-VPG graphs for each non-negative integer k. Finally, we show that every Chordal B_1-EPG graph is at the intersection of VPT and EPT.*

1. Introduction

EPG graphs were introduced by Golumbic, Lypshteyn, and Stern (2009) and consist of the intersection graphs of sets of paths on the orthogonal grid, whose intersections are taken considering the edges of the paths. If the intersections of the paths consider the vertices and not the edges, the resulting graph class is called VPG graphs.

The study of graphs whose host is a tree or a grid has motivation related to the problem of VLSI design that combines the notion of edge/vertex intersection graphs of paths in a tree/grid with a VLSI grid layout model, see [Golumbic et al. 2009]. The number of bends in an integrated circuit may increase the layout area, and consequently, increase the cost of chip manufacturing. This is one of the main applications that instigate research on the EPG/VPG representations of some graph families when there are constraints on the number of bends in the paths used in the representation. Other applications and details on circuit layout problems can be found in [Bandy and Sarrafzadeh 1990, Molitor 1991].

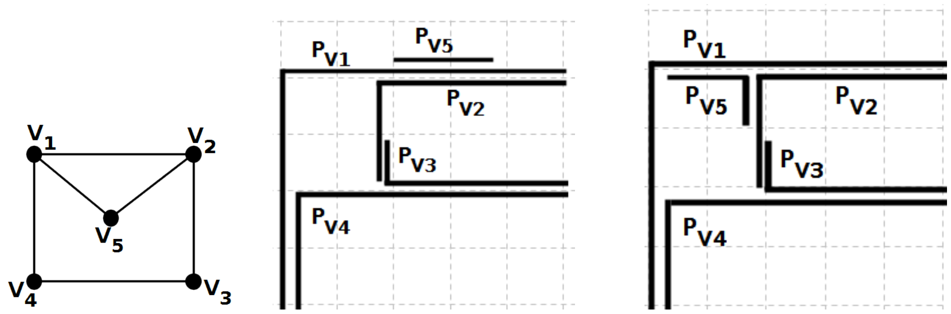
A graph is a B_k-EPG graph if it admits a representation in which each path has at most k bends. The *bend number* of a graph G is the smallest k for which G is a B_k-EPG graph. Analogously, the bend number of a class of graphs is the smallest k for which all graphs in the class have a B_k-EPG representation. Interval graphs have bend number 0, trees have bend number 1, and outerplanar graphs have bend number 2. The bend number for the class of planar graphs is still open, but it is either 3 or 4.

*Claudson F. Bornstein has been an extra oficial advisor.

The class of EPG graphs has been studied in several papers, such as [Alcón et al. 2016, Asinowski and Suk 2009, Cohen et al. 2014, Golumbic et al. 2009], among others. The investigations regarding EPG graphs frequently approach characterizations concerning the number of bends of the graph representations. Regarding the complexity of recognizing B_k-EPG graphs, only the complexity of recognizing a few of these sub-classes of EPG graphs have been determined: B_0-EPG graphs can be recognized in polynomial time, since it corresponds to the class of interval graphs; in contrast, recognizing B_1-EPG and B_2-EPG graphs are NP-complete problems. Also, note that the paths in a B_1-EPG representation have one of the following shapes: ⌞, ⌟, ⌜ and ⌝. Cameron et al. [Cameron et al. 2016] showed that for each non-empty $S \subset \{$⌞, ⌟, ⌜, ⌝$\}$, it is NP-complete to determine if a graph G has a B_1-EPG representation using only paths with shape in S.

A collection C of sets satisfies the Helly property when every sub-collection of C that is pairwise intersecting has at least one common element. The study of the Helly property is useful in several areas of science. We can enumerate applications in semantics, code theory, computational biology, database, graph theory, optimization, and linear programming, see [Dourado et al. 2009].

The Helly property can also be applied to B_k-EPG representations, where each path is considered as a set of edges. A graph G has a Helly-B_k-EPG representation if there is a B_k-EPG representation of G where each path has at most k bends, and this representation satisfies the Helly property. Figure 1(a) presents two B_1-EPG representations of a graph with five vertices. Figure 1(b) illustrates 3 pairwise intersecting paths $(P_{v_1}, P_{v_2}, P_{v_5})$, containing a common edge, so it is a Helly-B_1-EPG representation. In Figure 1(c), although the three paths are pairwise intersecting, there is no common edge in all three paths, and therefore they do not satisfy the Helly property.



(a) A graph with 5 vertices

(b) B_1-EPG representation that satisfies the Helly property

(c) B_1-EPG representation that does not satisfy the Helly property

Figure 1. A graph with 5 vertices in (a) and some single bend representations: Helly in (b) and not Helly in (c)

The Helly property related to EPG representations of graphs has been studied in [Golumbic et al. 2009, Golumbic et al. 2013].

Let $\mathcal{F}$ be a family of subsets of some universal set U, and $h \geq 2$ be an integer. Say that $\mathcal{F}$ is *h-intersecting* when every group of h sets of $\mathcal{F}$ intersect. The *core* of $\mathcal{F}$, denoted by $core(\mathcal{F})$, is the intersection of all sets of $\mathcal{F}$. The family $\mathcal{F}$ is *h-Helly* when every h-intersecting subfamily $\mathcal{F}'$ of $\mathcal{F}$ satisfies $core(\mathcal{F}') \neq \emptyset$. On the other hand, if for every subfamily $\mathcal{F}'$ of $\mathcal{F}$, there are h subsets whose core equals the core of $\mathcal{F}'$, then $\mathcal{F}$ is said to be *strong h-Helly*. Note that the Helly property that we will consider in this paper is precisely the property of being 2-Helly.

The *Helly number* of the family $\mathcal{F}$ is the least integer h, such that $\mathcal{F}$ is h-Helly. Similarly, the *strong Helly number* of $\mathcal{F}$ is the least h, for which $\mathcal{F}$ is strong h-Helly. It also follows that the strong Helly number of $\mathcal{F}$ is at least equal to its Helly number. In [Golumbic et al. 2009] and [Golumbic et al. 2013], they have determined the strong Helly number of B_1-EPG graphs.

In this thesis, we analyze structural characterizations and complexity aspects regarding B_k-EPG graphs. Our main focus is on the class of B_1-EPG graphs satisfying the Helly property, called Helly-B_1-EPG. We show that the problem of recognizing Helly-B_1-EPG graphs is NP-complete. Besides, other intersection graph classes such as VPG, EPT, and VPT were also studied. We completely solve the problem of determining the Helly and strong Helly numbers of B_k-EPG graphs and B_k-VPG graphs for each non-negative integer k. Finally, we show that every Chordal B_1-EPG graph is at the intersection of VPT and EPT.

Next, we present the list of papers, related to this thesis, developed during the doctoral research. Recall that in Theoretical Computer Science the list of authors is usually arranged in alphabetical order.

1. BORNSTEIN, C. F.; GOLUMBIC, M.C.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. The Complexity of Helly-B1-EPG graph Recognition. In: Discrete Mathematics & Theoretical Computer Science (DMTCS), vol. 22 no. 1, 2020.
2. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. Relationship Among B_1-EPG, VPT and EPT Graphs Classes. Accepted for publication in journal Discussiones Mathematicae Graph Theory (DMGT) on March 09, 2021.
3. BORNSTEIN, C. F.; MORGENSTERN, G.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. Helly and Strong Helly Numbers of B_k-EPG and B_k-VPG Graphs. Submitted to journal Discussiones Mathematicae Graph Theory (DMGT) on May 16, 2020.

The following are papers published in conferences:

1. BORNSTEIN, C. F.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. A Complexidade do Reconhecimento de Grafos B1-EPG-Helly. In: 50º SBPO - Simpósio Brasileiro de Pesquisa Operacional, 2018.
2. BORNSTEIN, C. F.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. Sobre a Dificuldade de Reconhecimento de Grafos B1-EPG-Helly. In: XXXVIII Congresso da Sociedade Brasileira de Computação, III Encontro de Teoria da Computação, 2018
3. BORNSTEIN, C. F.; SANTOS, T. D.; SOUZA, U. S.; SZWARCFITER, J. L. The complexity of B1-EPG-Helly graph recognition. In: VIII Latin American Workshop On Cliques in Graphs (LAWCG), ICM 2018 Satellite Event, 2018.
4. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. Identifying Subclasses of Helly-B_1-EPG Graphs. 52nd Brazilian Operational Research Symposium (SBPO), 2020.

5. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. On Subclasses of Helly-B_1-EPG Graphs. Reunión Anual de la Unión Matemática Argentina (virtUMA), 2020.
6. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. Paths on Hosts: B_1-EPG, EPT and VPT Graphs. Latin American Workshop on Cliques in Graphs (LAWCG), 2020.
7. ALCON, L.; MAZZOLENI, M. P.; SANTOS, T. D. On Helly-B_1-EPG Graphs. Submitted to Revista Matemática Contemporânea (SBM) on March 8, 2021.

The main results of this work are briefly presented as follows.

2. The Helly property and EPG graphs

First, we demonstrate that every graph is a Helly-EPG graph and we present some subclasses of B_1-EPG graphs that are incomparable with Helly-B_1 EPG. We present a characterization of Helly-B_1-EPG representations, and finally we demonstrate the NP-completness of the Helly-B_k EPG recognition problem.

The study starts with the following lemma.

Lemma 1 ([Golumbic et al. 2009])**.** *Every graph is an EPG graph.*

We show that this result extends to Helly-EPG graphs.

Lemma 2. *Every graph is a Helly-EPG graph.*

Corollary 3. *For every graph G containing μ maximal cliques, it holds that*

$$b_H(G) \leq \mu - 1.$$

Theorem 4. $[\llcorner] \subsetneq [\llcorner, \urcorner] \subsetneq$ *Helly-B_1 EPG, and Helly-B_1 EPG is incomparable with* $[\llcorner, \ulcorner]$ *and* $[\llcorner, \ulcorner, \urcorner]$.

Lemma 5. *A B_1-EPG representation of a graph G is Helly if and only if each clique of G is represented by an edge-clique, i.e., it does not contain any claw-clique.*

The HELLY-B_k EPG RECOGNITION problem can be formally described as follows.

	HELLY-B_k EPG RECOGNITION
Input:	A graph G and an integer $k \leq \lvert V(G)\rvert^c$, for some fixed c.
Goal:	Determine if there is a set of k-bend paths $\mathcal{P} = \{P_1, P_2, \ldots, P_n\}$ in a grid Q such that: • $u, v \in V(G)$ are adjacent in G if only if P_u, P_v share an edge in Q; and • $\mathcal{P}$ satisfies the Helly property.

At this point, it is important to note that the next result is non-trivial.

Theorem 6. HELLY-B_k EPG RECOGNITION *is in NP.*

Finally, we present our main result concerning the recognition of HELLY-B_1 EPG.

Theorem 7. HELLY-B_1 EPG RECOGNITION *is NP-complete.*

3. Helly and Strong Helly Numbers of B_k-EPG and B_k-VPG Graphs

In this section, we solve the problem for determining the Helly and strong Helly numbers, for both B_k-EPG and B_k-VPG graphs, for each non-negative integer k.

For EPG graphs, the Helly number of B_0-families is well known and is equal to 2, since B_0-EPG graphs coincide with interval graphs. It is also simple to conclude that the strong Helly number of B_0-EPG graphs are also equal to 2. For $k = 1$, we prove that both the Helly number and the strong Helly number of the class of B_1-families are equal to 3. For the class of B_2-families, we prove that these two parameters are equal to 4. The Helly and strong Helly number for B_3-families equal 8, and finally, these parameters are unbounded for $k \geq 4$.

As for VPG graphs, it is simple to verify that the Helly number of B_0-VPG graphs equals 2, and we prove that B_1-VPG have Helly number 4, B_2-VPG graphs have Helly number 6, B_3-VPG has Helly number 12, while the Helly number for B_4-VPG graphs is again unbounded.

Finally, the strong Helly number equals the Helly number of B_k-EPG graphs, for each k. Similarly, for B_k-VPG graphs.

As for existing results, Golumbic, Lipshteyn, and Stern [Golumbic et al. 2009] have already shown that the strong Helly number for B_1-EPG graphs equal 3, and for B_1-VPG graphs is equal to 4 (see [Golumbic and Morgenstern 2019], Theorem 11.13).

Theorem 8. *[Golumbic and Morgenstern 2019] Let P be a collection of single bend paths on a grid. If every two paths in P share at least one grid-edge, then P has strong Helly number 3. Otherwise, P has strong Helly number 4.*

To the best of our knowledgment, there is no other result concerning the strong Helly number or the Helly number of B_k-EPG graphs in the literature. However, for other classes, Golumbic and Jamison have determined the strong Helly number of the intersection of edge paths of a tree [Golumbic and Jamison 1985]. Finally, Asinowski, Cohen, Golumbic, Limouzy, Lipshteyn, and Stern have reported that the strong Helly number of B_0-VPG graphs equals two.

Table 1 summarizes the full classification regarding the strong Helly number and the Helly number of B_k-EPG graphs obtained in this thesis.

k	B_k-EPG	B_k-VPG
0	2	2
1	3	4
2	4	6
3	8	12
≥ 4	unbounded	unbounded

Table 1. Helly and Strong Helly Numbers for B_k-EPG and B_k-VPG Graphs

4. Relationship among B_1-EPG, EPT and VPT graph classes

We also have considered three different path-intersection graph classes: B_1-EPG, VPT and EPT graphs. We showed that $\{S_3, S_{3'}, S_{3''}, C_4\}$-free graphs and others non-trivial sub-classes of B_1-EPG graphs such as Bipartite, Block, Cactus and Line of Bipartite graphs are all Helly-B_1-EPG.

We presented an infinite family of forbidden induced subgraphs for the class B_1-EPG and in particular we proved that Chordal B_1-EPG $\subseteq$ VPT $\cap$ EPT.

Theorem 9. *Let G be a B_1-EPG graph. If G is $\{S_3, S_{3'}, S_{3''}, C_4\}$-free then G is a Helly-B_1-EPG graph.*

Theorem 10. *If G is a B_1-EPG and diamond-free graph then G is a Helly-B_1-EPG graph.*

Corollary 11. *If G is a Bipartite B_1-EPG graph then G is a Helly-B_1-EPG graph.*

Corollary 12. *Block, Cactus and Line of Bipartite graphs are Helly-B_1-EPG.*

Theorem 13. *Chordal B_1-EPG $\subsetneq$ VPT.*

Theorem 14. *Chordal B_1-EPG $\subsetneq$ EPT.*

References

Alcón, L., Bonomo, F., Durán, G., Gutierrez, M., Mazzoleni, M. P., Ries, B., and Valencia-Pabon, M. (2016). On the bend number of circular-arc graphs as edge intersection graphs of paths on a grid. *Discrete Applied Mathematics*, 234:12–21.

Asinowski, A. and Suk, A. (2009). Edge intersection graphs of systems of paths on a grid with a bounded number of bends. *Discrete Applied Math*, 157:3174–3180.

Bandy, M. and Sarrafzadeh, M. (1990). Stretching a knock-knee layout for multilayer wiring. *IEEE Transactions on Computers*, 39:148–151.

Cameron, K., Chaplick, S., and Hoàng, C. T. (2016). Edge intersection graphs of L-shaped paths in grids. *Discrete Applied Mathematics*, 210:185–194.

Cohen, E., Golumbic, M. C., and Ries, B. (2014). Characterizations of cographs as intersection graphs of paths on a grid. *Discrete Applied Mathematics*, 178:46–57.

Dourado, M. C., Protti, F., and Szwarcfiter, J. L. (2009). Complexity aspects of the Helly property: Graphs and hypergraphs. *The Electronic Journal of Combinatorics*, 17:1–53.

Golumbic, M. C. and Jamison, R. E. (1985). The edge intersection graphs of paths in a tree. *Journal of Combinatorial Theory*, B 38:8–22.

Golumbic, M. C., Lipshteyn, M., and Stern, M. (2009). Edge intersection graphs of single bend paths on a grid. *Networks*, 54:130–138.

Golumbic, M. C., Lipshteyn, M., and Stern, M. (2013). Single bend paths on a grid have strong Helly number 4. *Networks*, 62:161–163.

Golumbic, M. C. and Morgenstern, G. (2019). Edge intersection graphs of paths on a grid. In *50 years of Combinatorics, Graph Theory, and Computing*, pages 193–209. Chapman and Hall/CRC.

Molitor, P. (1991). A survey on wiring. *Journal of Information Processing and Cybernetics, EIK*, 27:3–19.