Connected and Disconnected Matchings

Bruno P. Masquio[1*],
Paulo E. D. Pinto[1] ,
Jayme L. Szwarcfiter[1,2]

[1]Universidade do Estado do Rio de Janeiro (UERJ) – Rio de Janeiro, RJ – Brazil

[2]Universidade Federal do Rio de Janeiro (UFRJ) – Rio de Janeiro, RJ – Brazil

{brunomasquio,pauloedp}@ime.uerj.br, jayme@nce.ufrj.br

Abstract. *Matching problems in graphs have been studied for a long time, achieving important results in both theoretical and practical aspects. Over the decades, many variations of matching problems and results were studied. Some of them can be solved in polynomial time, while others apparently cannot, unless* P = NP. *In this thesis, we briefly present the history of matching problems and their complexities, along with a survey of some of their variations, and their state-of-the-art. We also give new results on one of these variations:* $\mathscr{P}$*-matchings. A matching* M *is a* $\mathscr{P}$*-matching if the subgraph induced by the endpoints of the edges of* M *satisfies property* $\mathscr{P}$*. As examples, for appropriate choices of* $\mathscr{P}$*, the problems* INDUCED MATCHING, UNIQUELY RESTRICTED MATCHING, ACYCLIC MATCHING, CONNECTED MATCHING *and* DISCONNECTED MATCHING *arise. In this thesis, we focus our study on three:* DISCONNECTED MATCHING, CONNECTED MATCHING *and its weighted version,* WEIGHTED CONNECTED MATCHING. *To this end, we developed NP-completeness proofs, classical and parameterized complexity analysis, as well as exact polynomial algorithms, considering these problems in general and subject to some constraints.*

1. Introduction

A graph is a mathematical tool that represents elements as vertices and connections between them as edges. Due to its wide range of applications, some concepts emerge around them, such as matchings, which consist of a pairwise disjoint subset of the edges of a graph. Matchings modeling can lead to algorithms that solve theoretical or real-life problems. Some can be solved in a satisfactory time by a computer, while others apparently cannot; this is addressed in the standard study of computational complexity theory [Garey and Johnson 1979].

Many characterizations and applications in the subject of matchings have motivated scientists to expand the concept, considering some restrictions and generalizations. This thesis focuses on a restriction of matchings, known as $\mathscr{P}$-matchings, in which the subgraph induced by saturated vertices must satisfy property $\mathscr{P}$.

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Among the first to consider $\mathscr{P}$-matchings were Stockmeyer and Vazirani [Stockmeyer and Vazirani 1982]; they studied matchings in which edges have a certain distance between them. For a specific distance, those matchings yield subgraphs induced by its vertices that are 1-regular, which were later known as induced matchings.

Later, other types of matchings were defined and studied, such as uniquely restricted matchings [Golumbic et al. 2001], acyclic matchings [Goddard et al. 2005], disconnected matchings [Goddard et al. 2005], and connected matchings [Goddard et al. 2005]. The latter two, proposed by Goddard et al. [Goddard et al. 2005], contemplate connectivity properties of the corresponding subgraph. Some of them were also considered on edge-weighted graphs [Klemz and Rote 2022][Panda et al. 2020][Fürst and Rautenbach 2019].

1.1. Contributions

We begin this thesis with a vast survey on the history of matchings in graphs, with some of their main theorems and problems along with their state-of-the-art. Among them, we covered edge and vertex weighted matchings, minimum maximal matchings, counting all matchings in a graph, relations between edge colorings and matchings, removal of matchings from a graph, and $\mathscr{P}$-matchings. This survey itself serves as an updated reference of matching problems and their state-of-the-art. It can help future work on the subject, as one can familiarize with what is currently being discussed and also can give a starting point for upcoming research.

Our main results are on the $\mathscr{P}$-matchings. We explore $\mathscr{P}$-matchings for two properties: whether the subgraph induced by saturated vertices is connected or disconnected. The authors who considered these properties [Goddard et al. 2005] asked about the complexity of finding maximum disconnected matchings on a given graph; this problem remained open for 16 years and was answered by the results from this thesis. Our study showed not only that the problem is NP-complete, but also went further, by considering the number of connected components as a parameter of the problem. We also studied it for specific graph classes and parameterizations.

A different problem examined was CONNECTED MATCHING, as we improved the previous algorithm [Goddard et al. 2005] and analyzed the complexity of its version for weighted graphs, WEIGHTED CONNECTED MATCHING. We showed that this problem is NP-complete for several graph classes, while it can be solved in polynomial time for others. We also considered the parameterized approach, showing that the problem is FPT for the treewidth parameter, and does not admit a polynomial kernel when parameterized by the size of the vertex cover under theoretical complexity assumptions.

The results in this thesis are the current state-of-the-art for the problems considered and are summarized in Tables 1 and 2.

2. Disconnected Matchings

Our preliminary results indicate the complexity of finding disconnected matchings on graphs. In this kind of matching, the subgraph induced by the saturated vertices must have a minimum number of connected components. We studied it for various graph classes and constraints. The corresponding problem is defined as follows, and our results of this section are summarized in Table 1.

Graph class	c	Complexity
General	$c = 1$	Same as MAXIMUM MATCHING
Bipartite	Fixed $c \geq 2$	NP-complete
Chordal	Input	XP and NP-complete
Bounded degree	Input	NP-complete
Interval	Input	$\mathcal{O}(n^2 c \max\{nc, m\sqrt{n}\})$
Treewidth t	Input	$\mathcal{O}\big(8^t \eta_{t+1}^3 n^2\big)$

Table 1. Complexity results for DISCONNECTED MATCHING restricted to some input scopes. We denote by η_i the i-th Bell number.

DISCONNECTED MATCHING
Instance: A graph G and two integers k and c.
Question: Is there a matching M with at least k edges such that $G[M]$ has at least c connected components?

In the following, we give one of the most important results of the thesis, which is the NP-completeness for bipartite graphs of c-DISCONNECTED MATCHING, that is, the problem where c is a fixed constant.

Theorem 1. *For every $c \geq 2$, c-*DISCONNECTED MATCHING *is* NP-complete *on bipartite graphs of diameter 3 and general graphs of diameter 2.*

This negative result holds for bipartite graphs but not for every graph class. One example is the wide class of graphs whose minimal separators can be enumerated in polynomial time [Shen and Liang 1997] [Goddard et al. 2005], which includes chordal graphs. Nevertheless, for the latter class, we showed that if c is part of the input, the problem turns out to be NP-complete. In addition, with a little modification in the reduction, we can also prove it for graphs with bounded maximum degree.

Theorem 2. DISCONNECTED MATCHING *is* NP-complete *for chordal graphs with diameter* 2.
Theorem 3. DISCONNECTED MATCHING *is* NP-complete *even for graphs with maximum degree bounded by a constant and diameter* 2.

We showed that having c as part of the input instead of a fixed constant can turn the problem NP-hard. Yet, this pattern is not true for every graph class, as we show next.
Theorem 4. DISCONNECTED MATCHING *can be solved in polynomial time on interval graphs.*

Another property that can help the tractability of DISCONNECTED MATCHING is the treewidth of the input graph. This leads to the FPT result as follows.
Theorem 5. DISCONNECTED MATCHING *can be solved in* FPT *time when parameterized by treewidth.*

This theorem also implies tractability for several other parameters, such as vertex cover and max leaf number. Next, we provide kernelization lower bounds when parameterized by vertex cover and when parameterized by vertex deletion distance to clique.
Theorem 6. DISCONNECTED MATCHING *does not admit a polynomial kernel when jointly parameterized by vertex deletion distance to clique and number of edges in the matching unless* NP $\subseteq$ coNP/poly.

Graph class		Complexity	
		Weights ≥ 0	Any weights
General		NP-complete	
Bipartite having diameter at most 4		NP-complete	
Chordal		P	NP-complete
Starlike		P	NP-complete
Planar	bipartite	NP-complete	
Planar	subcubic	?	NP-complete
$\Delta \leq 2$		P	
Tree		P	

Table 2. Summary of our results for WEIGHTED CONNECTED MATCHING.

Our investigation on disconnected matchings concludes with the NP-hardness proof in determining whether the sizes of a maximum i-disconnected matching($\beta_{d,i}(G)$) and a maximum j-disconnected matching($\beta_{d,j}(G)$) are equal for an input graph G and fixed parameters i, j.

Theorem 7. *Given a graph G, deciding if $\beta_{d,i}(G) = \beta_{d,j}(G)$ is* NP-hard *even for diameter three bipartite graphs for every fixed i and j, $1 \leq i < j$.*

3. Connected Matchings

Next, we turn our attention to connected matchings, which is a $\mathscr{P}$-matching for the property of being connected. We studied its unweighted and weighted versions in various graph classes and weight constraints. The corresponding problems are defined as follows, and our results of this section are summarized in Table 2.

MAXIMUM CONNECTED MATCHING
Instance: A graph G.
Task: Find a connected matching M of G with maximum cardinality.

WEIGHTED CONNECTED MATCHING
Instance: An edge-weighted graph G and an integer k.
Question: Is there a connected matching M of weight at least k?

MAXIMUM WEIGHT CONNECTED MATCHING
Instance: An edge-weighted graph G.
Task: Find a connected matching M of G of maximum weight.

We begin our results for connected matching by showing that a maximum connected matching in a graph can be obtained in linear time, given an ordinary maximum matching. As an immediate result, we know that MAXIMUM CONNECTED MATCHING can be solved in the same time complexity as MAXIMUM MATCHING. We gave an algorithm based on the original proof that the problem in P [Goddard et al. 2005].

Theorem 8. *Given a maximum matching in an arbitrary graph G, a maximum connected matching in G can be found in linear time.*

3.1. Weighted Connected Matchings

Taking into account that weighted variations of matchings are widely studied, we decided to consider connected matchings under this approach. Besides, we see that some $\mathscr{P}$-matching concepts were already extended to edge-weighted problems, where, in addition to the matching to have a certain property $\mathscr{P}$, the sum of the weights of the matching edges must be sufficiently large [Klemz and Rote 2022][Panda et al. 2020][Fürst and Rautenbach 2019].

We begin our study on chordal graphs and one of their subclasses, starlike graphs. WEIGHTED CONNECTED MATCHING for these classes has different time complexities depending on the admittance of negative weights on the input graph; more specifically, we show that the problem is NP-complete if negative weights are allowed and is in P otherwise. To reach this result, we prove the NP-completeness for starlike graphs and provide a polynomial-time algorithm for chordal graphs having non-negative weights based on a polynomial-time reduction from MAXIMUM WEIGHT PERFECT MATCHING [Edmonds 1965, Duan et al. 2018].

Theorem 9. WEIGHTED CONNECTED MATCHING *is* NP-complete *even for starlike graphs whose edge weights are in* $\{-1, +1\}$.

Theorem 10. MAXIMUM WEIGHT CONNECTED MATCHING *for chordal graphs whose edge weights are all non-negative can be solved in polynomial time.*

The hardness result stated in Theorem 9 is highly dependent on the existence of several non-trivial cliques, which are forbidden in some graph classes, including bipartite graphs. This raises the question of whether or not the absence of these structures makes the problem easier; so, we answer this in the negative by showing that the problem remains hard on bipartite graphs having only binary weights.

Theorem 11. WEIGHTED CONNECTED MATCHING *is* NP-complete *on bipartite graphs of diameter* 4 *even if all edge weights are in* $\{0, 1\}$.

Theorem 12. WEIGHTED CONNECTED MATCHING *is* NP-complete *on planar bipartite graphs whose edge weights are in* $\{0, 1\}$.

Still on planar graphs but aside from the bipartite case, which is shown to be NP-complete, we investigate the complexity of WEIGHTED CONNECTED MATCHING in planar graphs under degree constraints, proving that the problem remains hard in the subcubic case.

Theorem 13. WEIGHTED CONNECTED MATCHING *is* NP-complete *even for subcubic planar graphs having edge weights in* $\{-1, +1\}$.

We have proven that WEIGHTED CONNECTED MATCHING is NP-complete even when some constraints are imposed, such as limits on the weights, planarity and degree bounds. But we also brought some tractable cases, presenting linear-time algorithms for graphs of maximum degree two, trees, and, more generally, a polynomial time algorithm for graphs of bounded treewidth.

Theorem 14. *Given a tree decomposition of width* t *of the* n*-vertex input graph,* MAXIMUM WEIGHT CONNECTED MATCHING *can be solved in* $2^{\mathcal{O}(t)}n^{\mathcal{O}(1)}$ *time.*

The fact that parameterizing by treewidth leads to an FPT algorithm immediately prompts an investigation into whether its decision version admits a polynomial kernel under the same parameterization. We answer this negatively with our last theorem.

Theorem 15. *Unless* $\mathsf{NP} \subseteq \mathsf{coNP/poly}$, WEIGHTED CONNECTED MATCHING *does not admit a polynomial kernel when parameterized by vertex cover number and required weight even if the input graph is bipartite and edge weights are in* $\{0, 1\}$.

4. Publications and conferences

We presented our work at several conferences and published a part of it in a journal during the Ph.D. period. Among them, we highlight COCOON 2021 and LATIN 2022, which have international reach, relatively low acceptance rates, and whose papers are published in the LNCS (Lecture Notes in Computer Science) series. We also presented abstracts and partial results at other conferences, as in Table 3.

The results for DISCONNECTED MATCHING problem, except for the equalities of matching numbers, and our algorithm for CONNECTED MATCHING were presented at the 27th International Computing and Combinatorics Conference (COCOON 2021) - whose acceptance rate was 42.75%. The article has 670 accesses so far. This paper was selected among those of high quality, invited and published [Gomes et al. 2023] in a special issue of the *Theoretical Computer Science* journal, which has an impact factor of 1.002 and was classified as *A1*(2013-2016) and *A2*(2017-2020) in *Qualis*.

In addition, our results for WEIGHTED CONNECTED MATCHING problem were presented at the 15th Latin American Theoretical Informatics Symposium (LATIN 2022) - whose acceptance rate was 40.4%. The article has 315 accesses so far. These results are part of a full paper, currently in arXiv [Gomes et al. 2022d] (200 reads in ResearchGate). Soon, it will be submitted to a journal. Some partial results on the same problem were also presented in VII Encontro de Teoria da Computaçao (ETC 2022) [Gomes et al. 2022a], where our article was selected among three of the best papers of the conference, receiving an honorable mention.

The thesis contains a result that was not covered in the previously mentioned articles, which is the complexity of deciding the equality of disconnected matching numbers. Soon, it will be part of a paper to be submitted to publication together with other results, such as the complexity of the problem comparing the sizes of a maximum disconnected matching and a maximum induced matching.

5. Conclusions

We believe this work gives a substantial contribution to current studies on matchings. Besides having a vast survey around matching problems and variants over the last decades, we brought some novel results. The result we developed concerning the complexity of disconnected matching, which was left open for 16 years, may be used to prove further results on other well-known problems related to it, such as induced matchings or uniquely restricted matchings. Our results for weighted connected matchings strengthen that the weighted approach on matching problems can be a determining factor in their complexities, even for some graph classes. Besides, the techniques used along the proofs can inspire other results alike or even in other fields, such as graph theory, combinatorial optimizations, and applications. Therefore, we are confident that the thesis not only represents

Conference	Reference
V Encontro de Teoria da Computação (ETC 2020)	[Masquio et al. 2020a]
Simpósio Brasileiro de Pesquisa Operacional (SBPO 2020)	[Masquio et al. 2020b]
Latin American Workshop on Cliques in Graphs (LAWCG 2020)	[Masquio et al. 2020c]
Congresso Nacional de Matemática Aplicada e Computacional (CNMAC 2021)	[Masquio et al. 2021]
VI Encontro de Teoria da Computação (ETC 2021)	[Gomes et al. 2021a]
27th International Computing and Combinatorics Conference (COCOON 2021)	[Gomes et al. 2021b]
VII Encontro de Teoria da Computação (ETC 2022)	[Gomes et al. 2022a]
10th Latin American Workshop on Cliques in Graphs (LAWCG 2022)	[Gomes et al. 2022b]
15th Latin American Theoretical Informatics Symposium (LATIN 2022)	[Gomes et al. 2022e]
XXI Latin-Iberoamerican Conference on Operations Research (CLAIO 2022)	[Gomes et al. 2022c]

Table 3. Conferences at which were presented partial results of the thesis.

the new state-of-the-art in the subject but also may serve as a reference and be useful for new methodologies and further results.

Possible directions for future work include studies related to $\mathscr{P}$-matchings, as there are many open questions on connected matchings and disconnected matchings.

For disconnected matchings, it can be considered in other graph classes, like ones with an exponential number of minimal separators for which DISCONNECTED MATCHING could be solved in polynomial time. Besides, we would like to know the complexity of obtaining a minimum maximal disconnected matching in a given graph. We are interested in the characterizations of graphs with the size of a minimum maximal disconnected matching. For its weighted version, someone could find if there are graph classes or weight constraints for which we could obtain maximum weight disconnected matchings in polynomial time.

Concerning connected matchings, we have strong results on its unweighted version, since we can obtain maximum connected matchings as fast as maximum matchings. However, minimum maximal connected matchings have not been studied yet. So, we want to know the complexity of obtaining a maximal connected matching of size at most k in a given graph. On its weighted version, other combinations of graph classes and allowed edge weights can be considered. Particularly, we would like to know the complexity of WEIGHTED CONNECTED MATCHING for diameter 3 bipartite graphs when weights are non-negative, chordal bipartite graphs, and subcubic planar graphs under the same constraint. Other graph classes of interest include cactus graphs and block graphs.

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