χ-Diperfect Digraphs

Caroline A. de Paula Silva[1,*], Orlando Lee (advisor)[1,†], Cândida N. da Silva (co-advisor)[2]
[1]Institute of Computing – University of Campinas (IC–UNICAMP))
Av. Albert Einstein, 1251, Cidade Universitária, CEP 13083-852, Campinas - SP - Brazil

[2]Department of Computing – Federal University of São Carlos (DComp–UFSCar)
Rod. João Leme dos Santos km 110 - SP-264, CEP 18052-780, Sorocaba - SP - Brazil

{caroline.silva,lee}@ic.unicamp.br, candida@ufscar.br

Abstract. *In 1982, Berge defined the class of χ-diperfect digraphs. A digraph D is χ-diperfect if for every induced subdigraph H of D and every minimum coloring $\mathcal{S}$ of H there exists a path P of H with exactly one vertex of each color class of $\mathcal{S}$. Berge also showed examples of non-χ-diperfect orientations of odd cycles and their complements. The ultimate goal in this research area is to obtain a characterization of χ-diperfect digraphs in terms of forbidden induced subdigraphs. In this work, we give steps towards this goal by presenting characterizations of orientations of odd cycles and their complements that are χ-diperfect. We also show that certain classes of digraphs are χ-diperfect. Moreover, we present minimal non-χ-diperfect digraphs that were unknown.*

1. Introduction

Let $G = (V(G), E(G))$ be a graph. We use standard concepts of path, cycle, complement of a graph and induced subgraph as defined in [Bondy and Murty 2008]. A set of vertices $S \subseteq V(G)$ is a *stable set* of G if, for any pair of vertices $u, v \in S$, the vertices u and v are non-adjacent. A set of vertices $T \subseteq V(G)$ is a *clique* of G if, for any pair of vertices $u, v \in T$, the vertices u and v are adjacent. A *(vertex)-coloring* $\mathcal{S}$ of a graph G is a partition of $V(G)$ into stable sets, also called *color classes*. We say that $\mathcal{S}$ is *minimum* if $\mathcal{S}$ has the smallest possible number of color classes.

Let $\chi(G)$ denote the *chromatic number* of G, that is, the number of color classes in a minimum coloring of G, and let $\omega(G)$ denote the number of vertices in a maximum clique of G. In any coloring, the vertices of a clique must receive distinct colors, so $\chi(G) \geq \omega(G)$. One may ask for which graphs this equality holds. Such a question led to the definition of an important class of graphs, named *perfect graphs*. We say that a graph G is *perfect* if, for every induced subgraph H of G, it follows that $\chi(H) = \omega(H)$.

Clearly, bipartite graphs (graphs that can be colored with two colors) are perfect. However, for any odd cycle C with at least five vertices, $\chi(C) = 3$ and $\omega(C) = 2$. In [Berge 1961], it was observed that complements of odd cycles with at least five vertices are not perfect (see Figure 1). Indeed, based on some empirical evidence, Berge conjectured that a graph is perfect if and only if its complement is perfect. This conjecture was proved in [Lovász 1972].

[*]This author was financed by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and FAPESP Proc. 2020/06116-4.

[†]This author was financed by CNPq Proc. 303766/2018-2, CNPq Proc 425340/2016-3 and FAPESP Proc. 2015/11937-9. ORCID: 0000-0003-4462-3325.

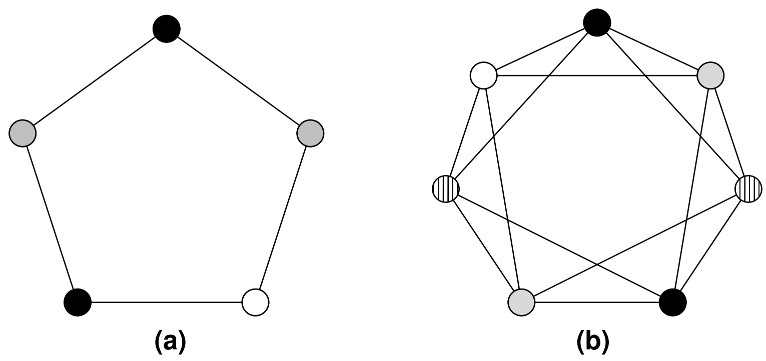

Figure 1. Examples of graphs that are not perfect. The graph G_1 depicted in (a) is a cycle with five vertices. Observe that $\chi(G_1) = 3$ and $\omega(G_1) = 2$. The graph G_2 depicted in (b) is the complement of a cycle with seven vertices. Observe that $\chi(G_2) = 4$ and $\omega(G_2) = 3$.

Theorem 1.1 ([Lovász 1972]). *A graph G is perfect if and only if its complement is perfect.*

Berge also proposed a stronger conjecture of a characterization of perfect graphs in terms of forbidden induced subgraphs. As mentioned before, if a graph G is perfect, then G does not contain an odd cycle with at least five vertices or its complement as an induced subgraph. Berge conjectured that the converse was true as well. This challenging conjecture attracted a lot of attention from the research community not only because of its structural appeal but also because of its algorithmic aspects. It was shown that many problems such as minimum coloring, maximum clique, and maximum stable set can be solved in polynomial time for perfect graphs [Grötschel et al. 1984]. This conjecture was eventually proved some decades later in [Chudnovsky et al. 2006] and it became known as the Strong Perfect Graph Theorem. This result is considered an important cornerstone in Graph Theory and the proof of [Chudnovsky et al. 2006] can be seen as the culmination of efforts of several researchers along many years. Further information about perfect graphs and their relevance can be found in [Chudnovsky et al. 2003].

Theorem 1.2 ([Chudnovsky et al. 2006]). *A graph G is perfect if and only if G does not contain an odd cycle with at least five vertices or its complement as an induced subgraph.*

In [Berge 1982], motivated by his research on perfect graphs and some well-known results for digraphs that relate colorings and paths, Berge introduced the concept of χ-diperfection of digraphs (defined later) which is the main theme of this research. In order to present such results and concepts, we need some definitions.

Let $D = (V(D), A(D))$ be a digraph. The *underlying graph* of D, denoted by $U(D)$, is the simple graph with vertex set $V(D)$ such that u and v are adjacent in $U(D)$ if and only if either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Similarly, we may obtain a digraph D from a graph G by replacing each edge uv of G by an arc (u, v), or an arc (v, u), or both; such digraph D is called a *super-orientation* of G. A super-orientation that does not contain a digon (a directed cycle of length two) is an *orientation*. A digraph D is *symmetric* if D is a super-orientation of a graph G in which every edge uv of G is replaced by both arcs (u, v) and (v, u).

If (u, v) is an arc of D, then we say that u *dominates* v and v is *dominated* by u. A *directed path* or *directed cycle* is an orientation of a path or cycle, respectively, in which each vertex dominates its successor in the sequence. Henceforth, when we say a path of a digraph, we mean a directed path (note that we do not use this convention for cycles). A path P is *hamiltonian* if $V(P) = V(D)$. When we say a cycle of a digraph, we mean either a super-orientation of an undirected cycle with at least three vertices or a digon.

A graph G is *complete* if $V(G)$ is a clique. A *semicomplete* digraph is a super-orientation of a complete graph. In [Rédei 1934] the following classical result was proved.

Theorem 1.3 ([Rédei 1934]). *Every semicomplete digraph has a hamiltonian path.*

A *stable set* of a digraph D is a stable set of its underlying graph $U(D)$. Similarly, a *coloring* of a digraph D is a coloring of its underlying graph $U(D)$. In [Roy 1967] and [Gallai 1968] independent proofs of a generalization of Rédei's Theorem were given. They showed that the number of vertices in a longest path of a digraph, denoted by $\lambda(D)$, is at least $\chi(D)$. Actually, they proved a stronger statement that uses the concept of *orthogonality*. A coloring $\mathcal{S}$ and a path P of D are *orthogonal* if P contains exactly one vertex of each color class of $\mathcal{S}$. We also say that P *is orthogonal to* $\mathcal{S}$ or vice versa.

Theorem 1.4 ([Gallai 1968][Roy 1967]). *Let D be a digraph. For every longest path P of D, there exists a coloring $\mathcal{C}$ of D such that P and $\mathcal{C}$ are orthogonal. In particular, $\lambda(D) \geq \chi(D)$.*

In [Berge 1982], it was noted that some digraphs have stronger properties in the relation between paths and colorings. In this context, Berge introduced the class χ-diperfect digraphs. A digraph D is *χ-diperfect* if for every minimum coloring $\mathcal{S}$ of D there is a path P orthogonal to $\mathcal{S}$, and this property holds for every induced subdigraph of D. Note that if a digraph D is χ-diperfect, then the inequality $\lambda(D) \geq \chi(D)$ (Gallai-Roy's Theorem) immediately holds.

Berge also proved that every symmetric digraph is χ-diperfect, as well as every digraph whose underlying graph is perfect. On the other hand, Berge showed the first examples of *obstructions* (minimal non-χ-diperfect digraphs). He showed that for a cycle of length five and for the complement of an odd cycle with at least five vertices, there are orientations that are not χ-diperfect. Examples given by Berge are depicted in Figure 2. The orientation of a cycle of length five in Figure 2a does not have a path with three vertices of all three colors because no path with three vertices contains vertex u, which belongs to a color class of size one. The digraph D on Figure 2b is a super-orientation of the complement of a cycle of length seven. We have $\chi(D) = 4$ and a 4-coloring of D, but D is not χ-diperfect because no path with four vertices contains vertex u, which belongs to a color class of size one.

Similarly to the problem of the perfect graphs, Berge was interested in obtaining a characterization of χ-diperfect digraphs in terms of forbidden induced subdigraphs. Due to the similarity of the problems and given the many decades between the proposal of Berge's Conjecture and the proof of the Strong Perfect Graph Theorem, it is reasonable to expect that finding a similar characterization for χ-diperfect digraphs may be a challenging problem. Moreover, to the best of our knowledge, this problem has not been much studied and many questions regarding a possible characterization of this class of digraphs remain open.

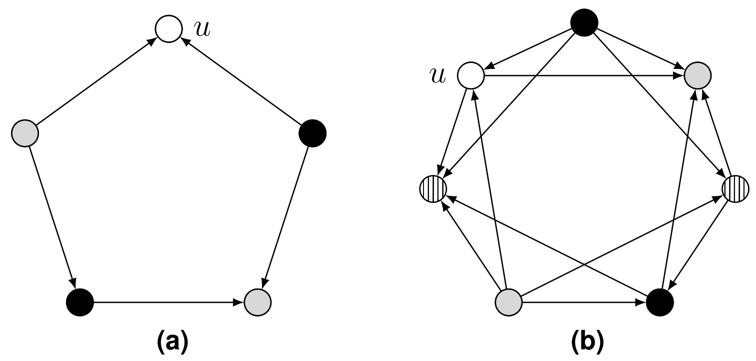

Figure 2. Examples of non-χ-diperfect digraphs given by Berge. The digraph depicted in (a) is an orientation of a cycle with five vertices. For the given minimum coloring, there is no path orthogonal to it. The digraph depicted in (b) is an orientation of the complement of a cycle with seven vertices. For the given minimum coloring, there is no path orthogonal to it.

In this paper, we present the approaches used and the results obtained in the first author's dissertation [de Paula Silva 2022]. In the first approach, we gave original characterizations of which super-orientations of odd cycles and which super-orientations of complements of odd cycles are χ-diperfect. Furthermore, we showed that some specific classes of digraphs that are free of such non-χ-diperfect structures are χ-diperfect. In the second approach, we investigated new obstructions. At first glance, it may be tempting to conjecture that the set of obstructions for χ-diperfect digraphs is exactly the set of non-χ-diperfect super-orientations of odd cycles and their complements. However, this is not true. We gave new examples of obstructions that show that forbidding such structures is not enough to characterize χ-diperfect digraphs.

2. Super-orientations of odd cycles and their complements

In this section, we present our results concerning the characterizations of super-orientations of odd cycles and their complements that are χ-diperfect. Before we present them, we need to introduce some notation. Let C_k denote the graph isomorphic to a cycle of length $k \geq 3$, and let $\overline{G}$ denote the *complement of* G. Let $\alpha(G)$ denote the *stability number* of G, that is, the number of vertices in a maximum stable set of a graph (or digraph) G.

Let D be a digraph. A vertex $v \in V(D)$ is a *source* if v is not dominated by any of its neighbors. Similarly, we say that v is a *sink* if v does not dominate any of its neighbors. Given a fixed minimum coloring of D, we say that a path P in D is a *rainbow path* if no two vertices of P are in the same color class; moreover, if $|V(P)| = k$, then we may say that P is a *k-rainbow path*.

2.1. Super-orientations of C_{2k+1}

Let D be a super-orientation of an odd cycle $C = (x_1, x_2, \ldots, x_{2\ell+1}, x_1)$ with at least five vertices. Let $P = (x_i, \ldots, x_j)$ be a subpath of C. We say that the subdigraph induced by $V(P)$ is a *sector* if each of x_i and x_j is a source or a sink in D; we say that the sector is *odd* if P has odd length, otherwise it is *even*. We say that D is a *conflicting odd cycle* if it contains at least two arc-disjoint odd sectors.

We proved that a super-orientation D of an odd cycle is χ-diperfect if and only if D is not a conflicting odd cycle. Figure 3 shows examples of conflicting odd cycles with minimum colorings that do not admit a 3-rainbow path. While for the digraphs depicted in Figure 2 there is a vertex in a color class of size one which does not belong to a path with $\chi(D)$ vertices, every vertex of both digraphs of Figure 3 belongs to a path with at least three vertices. Even so, both are non-χ-diperfect.

Theorem 2.1. *Let D be a super-orientation of an odd cycle. Then, D is χ-diperfect if and only if D is not a conflicting odd cycle.*

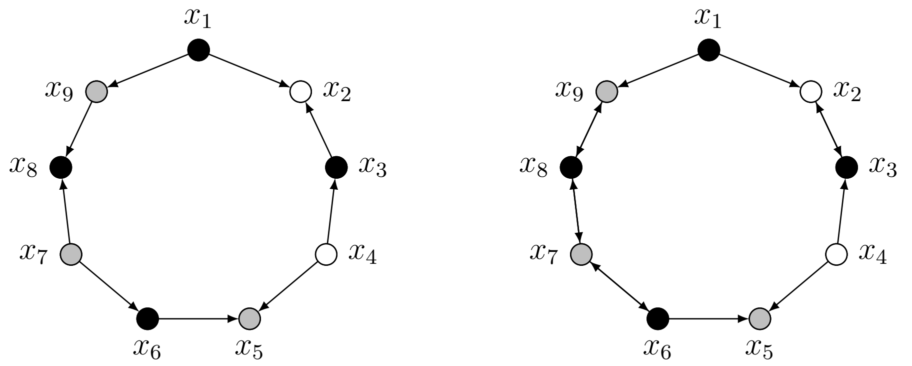

Figure 3. Minimum colorings of conflicting odd cycles which do not admit a 3-rainbow path. On the left, (x_1, x_2) and (x_4, x_5) are arc-disjoint odd sectors. On the right, (x_1, x_2, x_3, x_4) and (x_4, x_5) are arc-disjoint odd sectors.

2.2. Super-orientations of $\overline{C_{2k+1}}$

Let G be a graph isomorphic to a $\overline{C_{2k+1}}$, with $k \geq 2$. As cliques in G correspond to stable sets in $\overline{G}$ and vice versa, we deduce that $\alpha(G) = \omega(\overline{G}) = 2$ and $\omega(G) = \alpha(\overline{G}) = k$. Moreover, as we can partition the vertices of $\overline{G}$ into k cliques of size two and one clique of size one, we can verify that every minimum coloring of G must have exactly k color classes of size two and one color class of size one. Hence, $\chi(G) = k + 1$.

The characterization of non-χ-diperfect super-orientations of complements of odd cycles required several technical lemmas and auxiliary results. We describe the general idea behind the proof of such characterization, but we omit most of the technical details. We refer the reader to Lemmas 3.5 to 3.20 presented in [de Paula Silva 2022] which are important tools in the proof. These lemmas allow us to conclude that these digraphs must have the structure described next.

Let D be a super-orientation of a $\overline{C_{2k+1}}$, for $k \geq 2$, with a fixed minimum coloring $\mathcal{S}$. We can build a partition of D into two paths $P = (v_1, \ldots, v_k)$ and $Q = (w_1, \ldots, w_k)$ with k vertices and one single vertex u^* (which is the vertex in the only color class of size one of $\mathcal{S}$). Moreover, both $V(P)$ and $V(Q)$ induce semicomplete digraphs in D. Based on this partition, we show that D is non-χ-diperfect if and only if the following properties hold.

- there is no arc (v_j, v_i) or (w_j, v_i) for $j > i$,
- there is no arc (v_j, w_i) or (w_j, v_i) for $j \geq i$,
- there is an index p such that v_p and w_p are the non-neighbors of u^* in D,

- no vertex that precedes v_p and w_p in P and Q, respectively, are dominated by u^* and,
- no vertex that succeeds v_p and w_p in P and Q, respectively, dominates u^*.

Figure 4 shows an example of a non-χ-diperfect super-orientation of a $\overline{C_{11}}$ based on such structure. Although not all arcs are represented in the figure, recall that $V(P)$ and $V(Q)$ induce semicomplete digraphs.

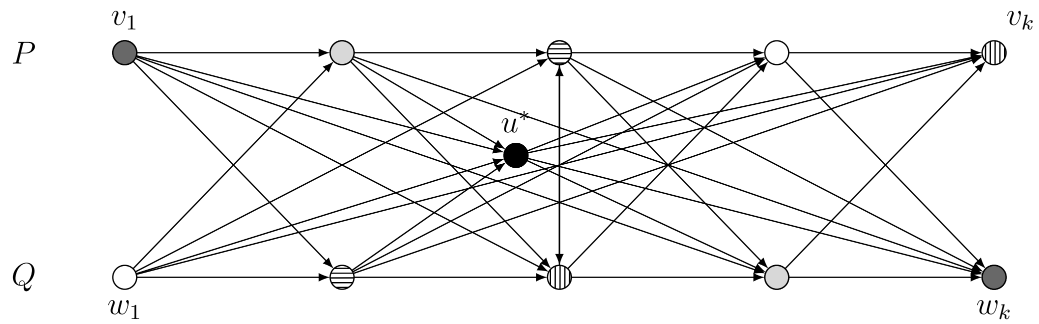

Figure 4. Non-χ-diperfect super-orientation of $\overline{C_{11}}$.

This structural characterization allowed us to prove another characterization with a more intuitive statement. In order to show that D is χ-diperfect, it suffices to ensure that every vertex of a super-orientation D of a $\overline{C_{2k+1}}$, with $k \geq 2$, belongs to a path with $\chi(D)$ vertices. Recall that this condition is not sufficient for a super-orientation of an odd cycle to be χ-diperfect (see Figure 3).

Theorem 2.2. *Let D be a super-orientation of $\overline{C_{2k+1}}$, with $k \geq 2$. Then, D is χ-diperfect if and only if every vertex of D belongs to a path with $k+1$ vertices.*

3. Generalizations of semicomplete digraphs

Rédei's Theorem [Rédei 1934] ensures the existence of a hamiltonian path in semicomplete digraphs. Because of that, it is easy to find rainbow paths in digraphs that contain semicomplete subdigraphs with $\chi(D)$ vertices. For example, all the digraphs whose underlying graph is perfect satisfy this property.

Based on these observations, we decided to study some classes of digraphs that are generalizations of semicomplete digraphs: locally in-semicomplete digraphs and locally arc in-semicomplete digraphs.

3.1. In-semicomplete digraphs

The classes of in-semicomplete and out-semicomplete digraphs were introduced in [Bang-Jensen 1990] and [Bang-Jensen 1995]. We say that a digraph D is *(locally) in-semicomplete* (respectively, *(locally) out-semicomplete*) if for every vertex $v \in V(D)$ the in-neighborhood (respectively, out-neighborhood) of v induces a semicomplete digraph. Note that if we reverse the orientation of each arc of an in-semicomplete digraph we obtain an out-semicomplete digraph and vice versa. Hence, all the results for in-semicomplete digraphs have analogous statements that hold for out-semicomplete digraphs.

Let D be an in-semicomplete digraph. Note that every induced subdigraph of D is also in-semicomplete. Let C be an induced cycle with at least four vertices of D. If a vertex of C has in-degree greater than one, then this implies the existence of a chord in C (an arc between non-consecutive vertices of C). So we conclude that C must be directed. Hence, C cannot contain conflicting odd cycles as induced subdigraphs. Moreover, we showed that D cannot contain a non-χ-diperfect super-orientation of a $\overline{C_{2k+1}}$ as an induced subdigraph. So, we do not have to deal with the obstructions described in Section 2.

We showed that every in-semicomplete digraph is χ-diperfect. The proof of this result relies on an ingenious induction hypothesis (see [de Paula Silva 2022, Lemma 4.5]). We remove the vertices of a color class and rather than assuming the existence of a rainbow path in the smaller digraph, we assume that there is a collection of rainbow paths with certain properties (described in the induction hypothesis). Then using some clever ideas we extend this collection to a collection of rainbow paths in the original digraph.

Theorem 3.1. *If D is an in-semicomplete digraph (out-semicomplete digraph), then D is χ-diperfect.*

3.2. Arc in-semicomplete digraphs

The classes of arc in-semicomplete digraphs and arc out-semicomplete digraphs were introduced in [Wang and Wang 2009]. A digraph D is *(locally) arc in-semicomplete* (respectively, *(locally) arc out-semicomplete*) if for every arc $(u, v) \in A(D)$ every in-neighbor (respectively, out-neighbor) of u and every in-neighbor (respectively, out-neighbor) of v are adjacent or coincide. As in the case of in-semicomplete digraphs, all the results for arc in-semicomplete digraphs have analogous statements that hold for arc out-semicomplete digraphs.

Let D be an arc in-semicomplete digraph. As shown in [Freitas and Lee 2021], every induced odd cycle with at least five vertices of D is directed and D cannot contain a super-orientation of the complement of an odd cycle with at least seven vertices as an induced subdigraph. Thus, here we do not have to deal with the obstructions of Section 2. Freitas and Lee also showed that, when D satisfies some additional properties, $V(D)$ may be partitioned into sets with a very particular structure. This result helped us to prove that every arc in-semicomplete digraph is χ-diperfect.

Theorem 3.2. *If D is an arc in-semicomplete digraph (arc out-semicomplete), then D is χ-diperfect.*

4. New minimal non-χ-diperfect digraphs

Recall that, by the Strong Perfect Graph Theorem, perfect graphs can be characterized by forbidding odd cycles with at least five vertices as well as their complements. Also, the examples of non-χ-diperfect digraphs given in [Berge 1982] when Berge introduced the χ-diperfect digraphs are precisely super-orientations of these structures. So it may be tempting to conjecture that the set of obstructions for χ-diperfect digraphs is exactly the set of non-χ-diperfect super-orientations of C_{2k+1} and $\overline{C_{2k+1}}$, for $k \geq 2$. However, such an assertion does not hold. After we obtained the characterizations presented in Section 2, we found more obstructions. Such digraphs have stability number two or three. All the obstructions and the method by which we found them are in [de Paula Silva 2022]

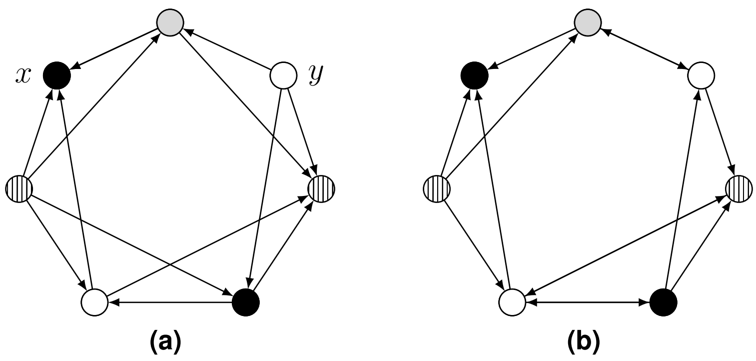

Figure 5. New χ-diperfect digraphs.

Figure 5 shows two examples of the new obstructions. The fact that these digraphs are obstructions is not obvious, so we refer the reader to [de Paula Silva 2022, Appendix A].

We observed that the underlying graph of the obstructions with stability number two that we have found is a spanning subgraph of a $\overline{C_{2k+1}}$, for $k \geq 3$. For example, if we add the arc joining x and y in the digraph depicted in Figure 5a, we obtain a super-orientation of a $\overline{C_7}$. Such observation led us to find a method to build an obstruction by deleting an arc from some non-χ-diperfect super-orientations of a $\overline{C_{2k+1}}$ for every $k \geq 3$.

Lemma 4.1. *For every $k \geq 3$, there is an obstruction that is obtained by deleting an arc from some non-χ-diperfect super-orientation of a $\overline{C_{2k+1}}$.*

Motivated by these observations, we decided to investigate digraphs with stability number two whose underlying graph does not contain spanning $(k+1)$-chromatic subgraphs of a $\overline{C_{2k+1}}$ with $k \geq 3$. In other words, we are interested in a family $\mathcal{H}$ of digraphs in which $D \in \mathcal{H}$ if and only if $\alpha(D) = 2$ and, for every induced subdigraph D' of D, the graph $U(D')$ is not a spanning $(k+1)$-chromatic subgraph of a $\overline{C_{2k+1}}$ with $k \geq 3$. One may note that this is equivalent to saying that a digraph $D \in \mathcal{H}$ if and only if every odd cycle of $\overline{U(D)}$ has length five.

The main result of this investigation is a characterization of χ-diperfect digraphs in $\mathcal{H}$. We proved that a digraph in $\mathcal{H}$ is χ-diperfect if and only if it does not contain an induced conflicting odd cycle. We omit this proof because it is not straightforward and it depends on several auxiliary results (see [de Paula Silva 2022, Section 5.2]).

Theorem 4.2. *Let D be a digraph in which every odd cycle of $\overline{U(D)}$ has length five. Then, D is χ-diperfect if and only if D does not contain a conflicting odd cycle as an induced subdigraph.*

5. Conclusion

In this paper, we presented the main results obtained in [de Paula Silva 2022]. These results were published in two paper. The first one was published in Discrete Mathematics [de Paula Silva et al. 2022a] and it contains the results concerning the characterizations of χ-diperfect super-orientations of odd cycles and their complements and the proofs that every in-semicomplete digraph and every arc in-semicomplete digraph is χ-diperfect. The second one was published in the proceedings of the 15th Latin American Symposium on Theoretical Informatics (LATIN 2022) [de Paula Silva et al. 2022b]. This paper contains the results concerning the new obstructions that we have found focusing on those with stability number two.

The new obstructions that we have found do not seem to have an evident pattern. At this moment, it seems hard to conjecture what would be the set of minimal non-χ-diperfect digraphs.

We note that our characterization of χ-diperfect super-orientations of complements of odd cycles led us to find an infinite family of counterexamples to a conjecture of Berge on the characterization of α-diperfect digraphs [de Paula Silva et al.]. The class of α-diperfect digraphs was introduced in [Berge 1982] simultaneously with the class of χ-diperfect digraphs with the aim of establishing a structural relation between stable sets and path-partitions on digraphs. This result was obtained shortly after the writing of de Paula Silva's master's dissertation.

6. Author's Contribution

The contribution of this research project went far beyond the initial expectations. When Caroline started her master's studies, we already knew the characterization of χ-diperfect super-orientations of odd cycles and just expected to find a characterization of χ-diperfect super-orientations of complements of odd cycles (or at least make progress towards that goal) as the main contribution of the project. Surprisingly, we found such characterization – and a proof that in-semicomplete digraphs are χ-diperfect – within the first six months of research. In the following two months, a proof that arc in-semicomplete digraphs are χ-diperfect was obtained, allowing us to submit a paper to an international journal by the end of the first year of research. This fast and steady progress was only possible due to Caroline's knowledge, creativity, research maturity and discipline above the average of a master's student. The proof that in-semicomplete digraphs are χ-diperfect can be seen as an example of Caroline's creativity and research maturity. She obtained this proof quickly and completely on her own in an unusual and ingenious way which surprised the advisors.

In the second year we decided to investigate whether the non-χ-diperfect super-orientations of odd cycles and complements of odd cycles found would be the only obstructions to χ-diperfectness. Again, Caroline's ideas were fundamental to obtain a negative answer. She started the process of identifying structural properties that an arbitrary obstruction must have, which lead us to finding new obstructions. Once again Caroline showed she is tailored for research for being able to ask good questions and attempting to answer them with creativity and perseverance. Another situation in which her researcher reasoning nature lead to important progress came right after finishing her master's studies. Aware of the analogue conjecture of Berge for α-diperfect digraphs and having acquired a deep understanding of the structure of super-orientations of complements of odd cycles, she realized that there was an infinite subset of such digraphs that were counterexamples to Berge's conjecture on α-diperfect digraphs proposed in 1982.

It must be emphasized that her master's studies occurred between March 2020 and March 2022 – the worst period of the COVID pandemic. During the first year of research, while making important progress, Caroline was also taking graduate courses for the first time in her academic life, and those were being adapted to online courses right after the strike of COVID pandemic.

References

Bang-Jensen, J. (1990). Locally semicomplete digraphs: a generalization of tournaments. *Journal of Graph Theory*, 14(3):371–390.

Bang-Jensen, J. (1995). Digraphs with the path-merging property. *Journal of Graph Theory*, 20(2):255–265.

Berge, C. (1961). Farbung von Graphen, deren samtliche bzw. deren ungerade Kreise starr sind. *Wissenschaftliche Zeitschrift*.

Berge, C. (1982). Diperfect graphs. *Combinatorica*, 2(3):213–222.

Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*. Springer.

Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R. (2006). The Strong Perfect Graph Theorem. *Annals of Mathematics*, 164:51–229.

Chudnovsky, M., Robertson, N., Seymour, P. D., and Thomas, R. (2003). Progress on perfect graphs. *Mathematical Programming*, 97:405–422.

de Paula Silva, C. A. (2022). χ-Diperfect Digraphs. Master's thesis, State University of Campinas - UNICAMP.

de Paula Silva, C. A., da Silva, C. N., and Lee, O. A family of counterexamples for a conjecture of Berge on α-diperfect digraphs. Submitted. `https://arxiv.org/abs/2207.08007`.

de Paula Silva, C. A., da Silva, C. N., and Lee, O. (2022a). χ-Diperfect digraphs. *Discrete Mathematics*, 345(9):112941.

de Paula Silva, C. A., da Silva, C. N., and Lee, O. (2022b). On χ-Diperfect Digraphs with Stability Number Two. In Castañeda, A. and Rodríguez-Henríquez, F., editors, *LATIN 2022: Theoretical Informatics*, pages 460–475, Cham. Springer International Publishing.

Freitas, L. I. B. and Lee, O. (2021). Some results on structure of all arc-locally (out) in-semicomplete digraphs. `https://arxiv.org/abs/2104.11019`.

Gallai, T. (1968). On directed paths and circuits. *Theory of graphs*, pages 115–118.

Grötschel, M., Lovász, L., and Schrijver, A. (1984). Polynomial algorithms for perfect graphs. In *North-Holland mathematics studies*, volume 88, pages 325–356. Elsevier.

Lovász, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2(3):253–267.

Rédei, L. (1934). Ein kombinatorischer satz. *Acta Litt. Szeged*, 7(39-43):97.

Roy, B. (1967). Nombre chromatique et plus longs chemins d'un graphe. *ESAIM: Mathematical Modelling and Numerical Analysis-Modélisation Mathématique et Analyse Numérique*, 1(5):129–132.

Wang, S. and Wang, R. (2009). The structure of strong arc-locally in-semicomplete digraphs. *Discrete Mathematics*, 309(23-24):6555–6562.