Decomposition by maximal cliques and forbidden subgraphs for path graphs

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Abstract. *In this work, we study some aspects of characterizations by forbiddance. In the general case, we investigate the existence of these characterizations in quasiordered sets, when certain special properties are required. In the case of graphs, we design an algorithm for decomposition by maximal clique separators, which is applicable in the search for forbidden subgraphs for some classes of path graphs. Finally, we apply known techniques, as well as a tool which is introduced in this work, to find an infinite family of forbidden subgraphs for the class of path graphs* UE.

Resumo. *Neste trabalho, estudamos alguns aspectos das caracterizações por proibição. No caso geral, investigamos a existência destas caracterizações em conjuntos quasiordenados, quando certas propriedades especiais são exigidas. No caso de grafos, nós elaboramos um algoritmo para decomposição por cliques maximais separadoras, que se aplica na busca de subgrafos proibidos para algumas classes de grafos de caminho. Finalmente, aplicamos técnicas já conhecidas, em conjunto com uma nova ferramenta introduzida neste trabalho, para encontrar uma família infinita de subgrafos proibidos para a classe de grafos de caminho* UE.

1. Introduction

Characterizations by forbidden subgraphs or forbidden minors are often found in Graph Theory. This is in part because these characterizations condense a lot of the structural properties of the characterized class into a concise and elegant statement. In many cases, these properties allow for the solution of problems which are hard in general, and sometimes they also allow for the creation of efficient algorithms, although sometimes only in an indirect manner. A famous example of this is the Graph Minor Theorem of N. Robertson and P. Seymour, which states that every class of graphs which is closed under minors can be characterized by a finite set of forbidden minors, cf. [Lovász 2006]. In particular, this implies that every such class has polynomial-time recognition. This theorem indicates that the minor relation is fundamentally different from, say, the relations of subgraph or induced subgraph, since these relations do not satisfy an analogous theorem.

The author's Master's dissertation contemplates three different areas of study, and is accordingly composed of three distinct parts, presented as different sections in this summary. For each theorem appearing in this summary, we indicate the number under which it appears in the dissertation. In the first part, motivated by the discussion of the preceding paragraph, a general abstract study of the existence of characterizations by forbiddance under certain requirements is initiated, and we obtain results both of characterization and of undecidability. In the second part, we elaborate an algorithm for decomposing graphs by maximal clique separators, which is a basic tool in the study of classes of graphs defined by forbiddance. This algorithm corrects one elaborated by R. Tarjan for the same end. Finally, in the third part of the dissertation, we present a study of characterizations by classes of path graphs by forbiddance, including a survey of known results, as well as the construction of an infinite family of forbidden graphs for UE graphs.

More precisely, as presented in further detail in Section 2, we investigate the existence of characterizations by forbiddance in quasiordered sets, when certain natural properties of *conciseness* are demanded. In particular, we give a necessary and sufficient condition for a qoset to have characterizations by forbidding sets which are *minimal* (in a precise sense, satisfied by finite graphs with the induced subgraph relation), and we show that the problem of determining whether a closed set of finite graphs can be characterized by forbidding a finite set of induced subgraphs, or of subgraphs, is undecidable. This implies that the problem of determining whether a closed set in a general qoset has a finite forbidding set is also undecidable. The main results of this section were presented at the *2011 European Conference on Combinatorics, Graph Theory and Applications* (*EuroComb'11*), in Budapest, Hungary, August 29th – September 2nd, 2011.

The proofs of the characterizations by forbiddance in Graph Theory are, in general, based on structural theorems satisfied by the class under consideration. The classes of *path graphs*, generic name given to the several classes of intersection graphs of paths in trees, deserve special mention in this discussion, since they have an extensive and uniform theoretical framework, composed of structural theorems which are naturally applicable to characterizations by forbiddance, cf. [Petito 2002]. However, even with such a vast array of structural theorems, the problem of finding such characterizations for these classes has still proven to be very challenging. This is corroborated by the relatively few path graph classes for which the complete list of forbidden subgraphs is known at this time [Panda 1999, Cerioli and Petito 2005, Tondato et al. 2005, Lévêque et al. 2009].

Some of the most important structural theorems satisfied by path graph classes are based on the *decomposition by maximal clique separators*, a particular instance of a general method for solving graph problems known as *graph decomposition*. As far as we know, the only known algorithm to decompose a graph by its maximal clique separators is the one sketched by R. Tarjan in [Tarjan 1985]. However, as we show in Section 3, this algorithm is not correct. In that section, we also propose a correct modification to that algorithm, retaining the same $O(nm)$ time complexity. The main results of that section have appeared as an extended abstract in Cerioli, M. R., Nobrega, H., and Viana. P. (2010), Decomposition by maxclique separators, *Matemática Contemporânea*, 39:69 – 76, after having been presented at the *4th Latin American Workshop on Cliques in Graphs* (*LawCliques'10*), in Petrópolis, Brazil, November 16th – 19th, 2010. They have also been submitted as a full article to *Discrete Mathematics*.

Finally, in Section 4, we introduce a graph construction which allows us to determine a significant part of the forbidden subgraphs for the UE class of path graphs. It is also shown that the family of forbidden subgraphs so obtained contains *all* of the forbidden subgraphs for UE graphs which satisfy a certain set of conditions. The main results of that section were presented at the *10th Cologne-Twente Workshop on graphs and combinatorial optimization* (*CTW2011*), in Frascati, Italy, June 14th – 16th, 2011.

2. Characterizations by nice forbidding sets

Let $\mathcal{Q} = (U, R)$ be a qoset, i.e., let R be a reflexive and transitive binary relation over the set U, which is assumed to be nonempty. Given $X \subseteq U$, $\mathsf{Forb}_R X$ denotes the set $\{y \in U : \nexists x \in X \text{ s.t. } xRy\}$, and X is a *forbidding set* of $\mathsf{Forb}_R X$. A set $Y \subseteq U$ is *characterizable by forbiddance* when there exists $X \subseteq U$ such that $Y = \mathsf{Forb}_R X$. In the most general case, it is a well-known result that X is characterizable by forbiddance iff X is closed under R. However, the forbidding set used in the proof of this fact, namely $\overline{X}$, can hardly be called *nice* from any point of view. In practice, when doing such characterizations one is always interested in forbidding sets which have some nice property of conciseness, and which might say more of the structure of the class in question.

For example, a set $X \subseteq U$ is a *minimal forbidding set* when, for every $x \in X$ and $y \in U$, if yRx and $y \neq x$, then $y \in \mathsf{Forb}_R X$. For finite graphs, this is equivalent to the statement that, for each $G \in X$ and each vertex v of G, we have $G \setminus v \in \mathsf{Forb}_R X$. A sufficient condition for guaranteeing minimal forbidding sets is given by well-foundedness. We say $\mathcal{Q}$ is *well-founded* when every nonempty subset X of U has a *minimal* element, i.e., an element $x \in X$ such that yRx does not hold for any $y \in X \setminus \{x\}$.

Theorem (Theorem 12)**.** *If $\mathcal{Q}$ is well-founded, then every set which is closed under R has a minimal forbidding set.*

Thus, every class of finite graphs which is closed under induced subgraphs has a minimal forbidding set. However, it is easy to see that well-foundedness is not a necessary condition for the existence of minimal forbidding sets. In order to state the desired weaker condition, we say $\mathcal{Q}$ is *well-founded on closed sets* when every nonempty and closed subset of U has a minimal element.

Theorem (Theorem 16)**.** *X has a minimal forbidding set iff X is closed and the qoset $(\overline{X}, R \cap \overline{X}^2)$ is well-founded on closed sets.*

Another notion of conciseness can be given by forbidding sets which are finite. As discussed previously, it is now a classic result that every class of finite graphs which is closed under minors has a characterization by a finite forbidding set. As can be easily seen, an analogous result does not hold for finite graphs with induced subgraphs. Therefore, it would be desirable to characterize those closed sets of this qoset that *do* have such characterizations. In this context, our main results are the following (note that, in order for these results to make sense, formally we assume that all of the objects in question are *computable*, i.e., they are presentable to a computer as input).

Theorem (Theorem 25). *Let $\mathcal{P} = (U, R)$ be a countable poset, such that for each $x \in U$, the set $\{y \in U : yRx\}$ is finite. The problem of determining whether a closed set of $\mathcal{P}$ has a finite forbidding set is undecidable.*

Corollary (Theorems 23 and 30). *The problems of determining whether a closed set of a qoset has a finite forbidding set, and of determining whether a set of finite graphs closed under induced subgraphs has a finite forbidding set, are undecidable.*

3. Decomposition by maximal clique separators

Procedures for decomposing graphs into smaller pieces often play a central role in graph theory. A particular type of graph decomposition which has found many interesting applications is that of decomposition by *clique separators*, and in this discussion, the classical algorithm in the literature is the one found in [Tarjan 1985]. Tarjan added a note at the end of his paper proposing a simple modification of his algorithm to find a decomposition by maximal clique (*maxclique*, for short) separators, and claimed this modified algorithm retained the same $O(nm)$ time complexity. This algorithm has been used, for example, to recognize some classes of path graphs [Monma and Wei 1986, Spinrad 2003].

An *elimination ordering* π of G is a bijection between $V(G)$ and $\{1, \ldots, n\}$. We say $u, v \in V(G)$ are *fillable* if they are nonadjacent and there exists a path $P = u, x_1, \ldots, x_k, v$ in G such that $\pi(x_i) < \min\{\pi(u), \pi(v)\}$, for all $i \in \{1, \ldots, k\}$. Together, such pairs form the set F_π of *fill-in edges created by* π. Furthermore, we say π is a *minimal elimination ordering* (m.e.o.) when there is no other elimination ordering π' of G such that $F_{\pi'} \subset F_\pi$. For each vertex v of G, we define $C_\pi(v) = \{u \in V : \pi(u) > \pi(v)$ and $uv \in E \cup F_\pi\}$.

Using these concepts, R. Tarjan developed an $O(nm)$ algorithm to decompose a given graph by clique separators. However, in the author's dissertation, a minimal counterxample is presented, showing that the proposed algorithm is not correct. Our main result of this section is Algorithm DMS, below, which is a correct modification of Tarjan's proposed algorithm, and preserves its $O(nm)$ time complexity.

Theorem (Theorems 61 and 63). *If G has a maxclique separator, then for any minimal ordering π of G, some decomposition step of Algorithm DMS separates G. Furthermore, if at some step Algorithm DMS separates G into G_1 and G_2, then G_1 is an atom.*

4. A partial characterization by forbidden subgraphs of edge path graphs

In this section, where we use the uniform notation introduced in [Monma and Wei 1986], we show how a large family of forbidden subgraphs for UE graphs can be obtained using a construction from the well-known class of 4-critical graphs.

The main tool which has been used in characterizing path graph classes by forbidden subgraphs is the *Separator Theorem*, due to Monma and Wei, which is based on the following concepts. If C is a maxclique whose removal disconnects G, then C *separates* G, and if V_i is the vertex set of a connected component of $G \setminus C$, then $G_i = G[V_i \cup C]$ is a *separated graph* of G by C. If $v \in C$ has a neighbor $v_i \in V_i$, then G_i is a *neighboring subgraph* of v. Two separated graphs G_i and G_j of G by C are *antipodal* when there exist maxcliques C_i, C_i', C_i'' of G_i and C_j, C_j', C_j'' of G_j, such that:

Input: A graph G.
Output: A decomposition of G by maxclique separators, if one exists.
Compute an m.e.o. π of G ;
foreach $v \in V$ **do** compute $C_\pi(v)$;
foreach $v \in V$ *in increasing order w.r.t.* π **do**
  if $C_\pi(v)$ *is a separating clique of* G **then**
    $A(v) \leftarrow$ the vertex set of the conn. comp. of $G \setminus C_\pi(v)$ containing v ;
    $B(v) \leftarrow V \setminus (A(v) \cup C_\pi(v))$;
    if $\exists A' \subset A(v)$ *s.t.* $C_\pi(v) \cup A'$ *is a maxclique of* G *and* $A' \neq \emptyset$ **then**
      $G_1 \leftarrow G[A(v) \cup (C_\pi(v) \cup A')]$;
      $G_2 \leftarrow G[B(v) \cup (C_\pi(v) \cup A')]$; *"type (i) decomposition step"*
      $G \leftarrow G_2$;
    else if $\exists B' \subset B(v)$ *s.t.* $C_\pi(v) \cup B'$ *is a maxclique of* G **then**
      $G_1 \leftarrow G[A(v) \cup (C_\pi(v) \cup B')]$;
      $G_2 \leftarrow G[B(v) \cup (C_\pi(v) \cup B')]$; *"type (ii) decomposition step"*
      $G \leftarrow G_2$;
    end
  end
end

Algorithm 1: Decomposition by Maxclique Separators, DMS

1. $C_i \cap C'_j \neq \emptyset$;
2. $C_i \cap C \not\supseteq C''_j \cap C$;
3. $C_j \cap C'_i \neq \emptyset$; and
4. $C_j \cap C \not\supseteq C''_i \cap C$.

The *antipodal graph* of G by C, denoted by $\mathscr{A}(G, C)$, is the graph which has the separated graphs of G by C as vertices, and such that two vertices G_i and G_j are adjacent iff G_i and G_j are antipodal.

Although the class UE itself is not characterized by Monma and Wei's Separator Theorem, the subclass of UE graphs which are also chordal, denoted by UEC, is:

Theorem (Separator Theorem for UEC, Theorem 75)**.** *Let G be separated by a maxclique C. Then G is a* UEC *graph iff each separated graph G_i is a* UEC *graph, and $\mathscr{A}(G, C)$ has a 3-coloring in which the set of neighboring subgraphs of each $v \in C$ is 2-colored.*

The construction and the main result

Given G, construct the graph $\mathsf{constr}(G)$ by first subdividing each edge by a new vertex, and then transforming the set of new vertices used in these subdivisions into a clique C_G. It is not hard to see that if $\delta(G) \geq 2$, then C_G is the only separating maxclique of $\mathsf{constr}(G)$, and $\mathscr{A}(\mathsf{constr}(G), C_G) = G$.

For the main result, we need the following concepts. A graph G is *k-critical* when $\chi(G) = k$ and, for each proper subgraph H of G, we have $\chi(H) < k$. We denote by $\mathscr{C}_4$ the set of all graphs which are $\mathsf{constr}(G)$ for some 4-critical graph G.

Theorem (Theorem 100)**.** *$G \in \mathscr{C}_4$ if, and only if, G is a forbidden subgraph for* UEC *graphs, but not for* UV *graphs, such that G is separated by a maxclique C, with $\mathscr{A}(G, C)$ not 3-colorable, and such that each $v \in C$ has at most two neighboring subgraphs.*

In particular, $\mathscr{C}_4$ is a family of forbidden subgraphs for UEC, and since $\mathsf{constr}(G)$ is chordal for every G, we also have that $\mathscr{C}_4$ is a family of forbidden subgraphs for UE graphs. Furthermore, it follows that recognizing 4-critical graphs is as hard as recognizing this subfamily of forbidden subgraphs for UE.

5. Future work

As future research, we can highlight:

- Investigation of the relationship between the complexity of recognizing a set which is definable by forbiddance, and properties satisfied by its forbidding set.
- Application of the decomposition by maxclique separators to characterizations of other classes of graphs by forbiddance, and also apply them to the solution of other graph problems.
- Development of tools that enable one to find the complete characterizations of the various classes of path graphs by forbiddance. In particular, the search for a good characterization of the antipodality of perfect graphs, in the same mold as the existing ones for split and chordal graphs, and the search for properties of the antipodal graphs of forbidden subgraphs for UE that allow one to further determine the list of forbidden subgraphs for that class, especially for the cases not covered by Theorem 100.

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