Graph colorings and digraph subdivisions

Phablo F. S. Moura, Yoshiko Wakabayashi (advisor)[1]

[1]Instituto de Matemática e Estatística – Universidade de São Paulo (USP)
Rua do Matão 1010 – São Paulo – SP – Brazil

{phablo,yw}@ime.usp.br

Abstract. *This paper presents our studies on three vertex coloring problems on graphs and on a problem concerning subdivision of digraphs. Given an arbitrarily colored graph G, the convex recoloring problem consists in finding a (re)coloring that minimizes the number of color changes and such that each color class induces a connected subgraph of G. This problem is motivated by its application in the study of phylogenetic trees in Bioinformatics. In the k-fold coloring problem one wishes to cover the vertices of a graph by a minimum number of stable sets in such a way that every vertex is covered by at least k (possibly identical) sets. The proper orientation problem consists in orienting the edges of a graph so that adjacent vertices have different in-degrees and the maximum in-degree is minimized. Our contributions in these problems are in terms of algorithms, hardness, and polyhedral studies. Finally, we investigate a long-standing conjecture of Mader on subdivision of digraphs: for every acyclic digraph H, there exists an integer $f(H)$ such that every digraph with minimum out-degree at least $f(H)$ contains a subdivision of H as a subdigraph. We give evidences for this conjecture by proving it holds for classes of acyclic digraphs.*

1. Introduction

We next present our studies on three coloring problems on graphs and on a problem related to a long-standing conjecture about subdivision of digraphs. These studies form the PhD Thesis of Phablo F. S. Moura defended in 2017 at Universidade de São Paulo [Moura 2017]. Owing to space limitation, we only mention the main results without the proofs. We refer the reader to [Moura 2017] for more details.

Firstly, we focus on the coloring problems. Given an arbitrarily colored graph G with positive weights assigned to its vertices, the *convex recoloring problem* consists in finding a minimum weight (re)coloring such that each color class induces a connected subgraph of G. We also address a generalization of the classic vertex coloring problem. The *k-fold coloring problem* consists in covering the set of vertices of a graph into a minimum number of stable sets in such a way that every vertex is covered by at least k (possibly identical) stable sets. Note that, in the context of k-fold colorings, a coloring can be viewed as a *multiset* of stable sets. The third coloring problem studied is associated with oriented graphs. The *proper orientation problem* consists in orienting the edge set of a given graph so that adjacent vertices have different in-degrees and the maximum in-degree is minimized. Clearly, the in-degrees induce a partition of the vertex set into stable sets, that is, a coloring (in the conventional sense) of the vertices.

In addition to the mentioned coloring problems on graphs, we study a problem related to subdivision of *directed* graphs (or simply, *digraphs*). Given a graph G, the *subdivision of an edge* uv in G is a graph operation in which uv is removed from G, a new vertex, say w, and edges wu and wv are created. One may easily extend the concept of subdivision to digraphs. Given a digraph D, the *subdivision of an arc* uv in D is a digraph operation in which uv is removed from D, a new vertex w, and arcs uw and wv are created. Note that the directions of arcs uw and wv are coherent to the direction of uv. The graph (respectively, digraph) obtained from any sequence of subdivision operations is called a *subdivision* of G (respectively, of D).

In the eighties, Mader conjectured that, for every acyclic digraph H, there exists an integer $f(H)$ such that every digraph with minimum out-degree at least $f(H)$ contains a subdivision of H as a subdigraph. In this context, *the subdivision problem* consists in showing the existence of such a function for subclasses of acyclic digraphs.

During our studies we addressed the mentioned problems in terms of algorithms, computational complexity, polyhedral combinatorics, and structural graph theory. As we shall see, the contributions of the thesis are spread over these areas. In the following sections, for each problem, we present the definitions, known results and main contributions.

2. Convex recoloring

A coloring of the vertices of a connected graph is r-convex if each color class induces a subgraph with at most r components. We address the r-convex recoloring problem (r-CR) defined as follows. Given a graph G and a coloring of its vertices, recolor a minimum number of vertices of G so that the resulting coloring is r-convex. One may naturally define a weighted version of r-CR by assigning nonnegative weights to the vertices.

The 1-CR (or simply CR) problem has received considerable attention in the last years. Kanj and Kratsch [Kanj and Kratsch 2009] proved that this problem is $\mathcal{NP}$-hard for paths even if each color appears at most twice. Campêlo et al. [Campêlo et al. 2014] showed that the unweighted CR problem is $\mathcal{NP}$-hard on 2-colored grids. Very recently, Bar-Yehuda et al. [Bar-Yehuda et al. 2016] designed a $3/2$-approximation algorithm for general graphs in which each color appears at most twice.

The 1-CR problem was firstly investigated by Moran and Snir in 2005, motivated by its application in the study of phylogenetic trees. Given a set of species, a *phylogenetic tree* is a colored tree representing the course of evolution of these species. Its leaves correspond to the given set of species and its internal vertices correspond to extinct species. The colors of the vertices correspond to character states, where a *character* (coloring) is a biological attribute shared among all the species, and a *character state* (color) is the state of this character shown by a *species* (vertex). The more general concept of r-convexity, for $r \geq 2$, was proposed later, and it is also of interest in the study of protein-protein interaction networks and phylogenetic networks.

Our aim in the study of convex recoloring problems is twofold. On the theoretical side, we prove inapproximability and $\mathcal{W}[2]$-hardness results which provide a better understanding of the difficulty associated with the various convex recoloring problems that have been treated in the literature. More specifically, we proved the following theorems.

Theorem 2.1. *For every r and $k \geq 2$, and $\varepsilon < 1$, there is no $n^{1-\varepsilon}$-approximation for the* CR *problem on k-colored n-vertex bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.*

Theorem 2.2. *Let $f : \mathbb{N} \to \mathbb{Q}$ be a function of the form $f(x) = 2^{\text{poly}(x)}$. For every r and $k \geq 2$, there is no $f(n)$-approximation for the weighted r-*CR *problem on k-colored n-vertex bipartite graphs, unless $\mathcal{P} = \mathcal{NP}$.*

On the applied side, we propose an integer linear formulation for CR on general graphs and we design an algorithm to tackle the CR problem on trees. We provide computational experiments on instances that come from an application on phylogenetic trees and show that our solution method (on input instances with many colors) performs better than the best solving method described in the literature so far. We refer the reader to the thesis for more details concerning the formulations and computational experiments.

Some results concerning the inapproximability and parameterized complexity were published in an extended abstract [Moura and Wakabayashi 2017]. A full paper containing these results was submitted to a journal.

3. k-Fold coloring

We also address a natural generalization of the classic vertex coloring problem, namely the k-fold coloring problem, which was introduced in the Seventies. A k-fold x-coloring of a graph G is an assignment of (at least) k distinct colors from the set $\{1, 2, \ldots, x\}$ to each vertex such that any two adjacent vertices are assigned disjoint sets of colors. The k-th chromatic number of G, denoted by $\chi_k(G)$, is the smallest x such that G admits a k-fold x-coloring. Clearly, $\chi_1(G) = \chi(G)$ is the conventional chromatic number of G.

In the thesis, we present a dense theoretical study of the polytope associated with an integer linear programming formulation which is based on the concept of class representatives. We introduce an integer linear programming formulation (ILP) to determine $\chi_k(G)$ and study the facial structure of the corresponding polytope $\mathcal{P}_k(G)$. We present facets that $\mathcal{P}_{k+1}(G)$ inherits from $\mathcal{P}_k(G)$ and show how to lift facets from $\mathcal{P}_k(G)$ to $\mathcal{P}_{k+\ell}(G)$, for any $\ell \in \mathbb{Z}_>$. We project facets of $\mathcal{P}_1(G \circ K_k)$ into facets of $\mathcal{P}_k(G)$, where $G \circ K_k$ denotes the lexicographic product of G by a clique with k vertices. In both cases, we can obtain facet-defining inequalities from many of those known for the 1-fold coloring polytope. We also derive facets of $\mathcal{P}_k(G)$ from facets of stable set polytopes of subgraphs of G. In addition, we present classes of facet-defining inequalities based on strongly χ_k-critical webs and antiwebs, which are structures that play an important role in the description of stable set and coloring polytopes. We introduce this criticality concept and characterize the webs and antiwebs having such a property. These results extend and generalize known results for 1-fold coloring.

An extended abstract with some preliminary results was published in a conference [Campêlo et al. 2013]. A full paper containing our contributions concerning the k-fold coloring problem was published in a journal [Campêlo et al. 2016].

4. Proper orientation

An *orientation* of a graph G is a digraph D obtained from G by replacing each edge by exactly one of the two possible arcs with the same endpoints. For each $v \in V(G)$, the *in-degree* of v in D, denoted by $d^-_D(v)$, is the number of arcs with head v in D. An orientation D of G is *proper* if $d^-_D(u) \neq d^-_D(v)$, for all $uv \in E(G)$. The *proper orientation number* of a graph G, denoted by $\overrightarrow{\chi}(G)$, is the minimum of the maximum in-degree over all its proper orientations.

This graph parameter was introduced by Ahadi and Dehghan in [Ahadi and Dehghan 2013]. It is well-defined for any graph G since one can always obtain a proper $\Delta(G)$-orientation using the following procedure. Consider a vertex, say v, of maximum degree in G, orient all edges incident to v towards it, and repeat this procedure on $G - v$ if it is not an empty graph. Hence, it holds that $\overrightarrow{\chi}(G) \leq \Delta(G)$. Note that every proper orientation of a graph G induces a proper vertex coloring of G. Furthermore, observe that, in any k-orientation of G, $d^-(v) \in \{0, 1, \ldots, k\}$ for every $v \in V(G)$. Thus, it follows that $\overrightarrow{\chi}(G) \geq \chi(G) - 1$. Therefore, we have the following sequence of inequalities: $\omega(G) - 1 \leq \chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G)$. These inequalities are best possible in the sense that, for a complete graph K, $\omega(K) - 1 = \chi(K) - 1 = \overrightarrow{\chi}(K) = \Delta(K)$.

In their seminal paper [Ahadi and Dehghan 2013], Ahadi and Dehghan focused on algorithmic aspects of the proper orientation number of regular graphs and planar graphs. They proved that it is $\mathcal{NP}$-complete to decide whether $\overrightarrow{\chi}(G) = 2$ for planar graphs G. Additionally, they showed that computing the proper orientation number of 4-regular graphs is $\mathcal{NP}$-hard.

The proper orientation problem has also been studied from a structural point of view. Very recently, Araujo et al. [Araujo et al. 2016] showed that every cactus admits a proper orientation with maximum in-degree at most 7. Moreover, they proved that if G is a planar claw-free graph, then $\Delta(G) \leq 6$, which trivially implies $\overrightarrow{\chi}(G) \leq 6$. Also recently, Knox et al. [Knox et al. 2016] proved that 3-connected planar bipartite graphs have proper orientation number at most 6

Our contributions on the proper orientation problem are twofold: algorithmic and structural. From the structural point of view, we prove bounds for the parameter $\overrightarrow{\chi}$ on trees and general bipartite graphs. From the algorithmic point of view, we study the computational complexity of computing the proper orientation number of bipartite and bounded degree graphs. We now mention some of our main contributions on this topic.

Theorem 4.1. *If G is a bipartite graph, then $\overrightarrow{\chi}(G) \leq \left\lfloor \frac{\Delta(G)+\sqrt{\Delta(G)}}{2} \right\rfloor + 1$.*

Theorem 4.2. *If T is a tree, then $\overrightarrow{\chi}(T) \leq 4$.*

Theorem 4.3. *The following problem is $\mathcal{NP}$-complete:*
INPUT : *A planar bipartite graph G with $\Delta(G) = 5$.*
QUESTION : *$\overrightarrow{\chi}(G) \leq 3$?*

A full paper containing our contributions on the proper orientation problem was published in a journal [Araujo et al. 2015].

5. Subdivision of digraphs

A *subdivision of a graph G* is a graph obtained from G by replacing some of its edges by internally vertex-disjoint paths, that is, in a subdivision of G, for each $uv \in E(G)$, edge uv is replaced by a path P_{uv} with endpoints u and v, and newly created internal vertices. In the world of digraphs, arcs are replaced by directed paths. More precisely, a *subdivision of a digraph D* is a digraph obtained from D by replacing some of its arcs by internally vertex-disjoint directed paths with the same endpoints and oriented in the same direction as the corresponding arcs. Given two (di)graphs H and G, we say that G *contains a subdivision of H* if there exists a subgraph (respectively, subdigraph) G' of G such that G' is isomorphic to some subdivision of H.

Let $k \in \mathbb{N}$. We denote by K_k the *complete (undirected) graph* on k vertices. The *complete digraph* on k vertices, denoted by $\vec{K}_k$, is obtained from K_k replacing every edge of it by two arcs with the same endpoints and opposite directions, that is, every edge of K_k is replaced by a copy of $\vec{C}_2$, the directed cycle on 2 vertices. A *tournament* on k vertices is an orientation of the complete graph K_k. The *transitive tournament* on k vertices, denoted by TT_k, is a tournament on k vertices with no directed cycle. In the sixties, Mader [Mader 1967] established the following theorem for (undirected) graphs.

Theorem 5.1 (Mader [Mader 1967])**.** *For every positive integer k, there exists an integer $f(k)$ such that every graph with minimum degree at least $f(k)$ contains a subdivision of K_k, the complete graph on k vertices.*

Similarly, it would be interesting to find analogous results for digraphs. However, the obvious analogue that a digraph with sufficiently large minimum in-degree and minimum out-degree contains a subdivision of the complete digraph of order n is false as shown by Mader [Mader 1985].

Let γ be a digraph parameter. A digraph F is γ-maderian if there exists a least integer $\mathrm{mader}_\gamma(F)$ such that every digraph D with $\gamma(D) \geq \mathrm{mader}_\gamma(F)$ contains a subdivision of F. For a digraph D, $\delta^+(D)$ (respectively, $\delta^-(D)$) denotes the minimum out-degree (respectively, in-degree) and $\delta^0(D) = \min\{\delta^+(D), \delta^-(D)\}$. A natural question is to ask what digraphs F are *δ^+-maderian* (respectively, *δ^0-maderian*). Observe that every δ^+-maderian digraph is also δ^0-maderian and that $\mathrm{mader}_{\delta^+} \geq \mathrm{mader}_{\delta^0}$.

On the positive side, Mader conjectured that every acyclic digraph is δ^+-maderian. Since every acyclic digraph is the subdigraph of the transitive tournament on the same order, it is enough to prove that transitive tournaments are δ^+-maderian.

Conjecture 5.2 (Mader [Mader 1985])**.** *For every positive integer k, there exists a least integer $\mathrm{mader}_{\delta^+}(TT_k)$ such that every digraph D with $\delta^+(D) \geq \mathrm{mader}_{\delta^+}(TT_k)$ contains a subdivision of TT_k.*

Mader proved that $\mathrm{mader}_{\delta^+}(TT_4) = 3$, but even the existence of $\mathrm{mader}_{\delta^+}(TT_5)$ is still open. Given the remarkable difficulty of this conjecture, it is natural to consider subclasses of acyclic digraphs. An *in-arborescence* is an oriented tree in which all arcs are directed towards a vertex called *root*. In [Moura 2017], we give new evidences for Conjecture 5.2 by proving that all in-arborescences are δ^+-maderian.

6. Concluding remarks

The problems mentioned in this work have brought quite different challenges during our studies. Consequently, we had to address them also using different techniques, namely algorithmic, structural and polyhedral. Therefore, the contributions in the thesis are spread over these areas.

Despite the deep research carried out on those problems, there are still several interesting open questions that may lead to further (theoretical and applied) research.

Question 6.1. *Is our solving method for computing the convex recoloring efficient for solving real-world instances from Bioinformatics?*

Question 6.2. *Can the facet-defining inequalities and the representatives formulation be used in practice to compute k-fold colorings?*

Question 6.3. *Does there exist a constant k such that $\vec{\chi}(G) \leq k$ for planar graphs G?*

Question 6.4. *Does the Mader's Conjecture 5.2 hold for general oriented trees?*

To conclude, we remark that the results in the thesis [Moura 2017] have been published in the *Theoretical Computer Science* [Araujo et al. 2015] and *Discrete Optimization* [Campêlo et al. 2016]. Additionally, we have submitted two full papers to journals: one containing the results on the convex recoloring problem and the other containing the results about subdivisions of digraphs. We have also presented preliminary results in international conferences as *ICGT 2014* and *LAGOS* [Campêlo et al. 2013, Moura and Wakabayashi 2017].

References

Ahadi, A. and Dehghan, A. (2013). The complexity of the proper orientation number. *Information Processing Letters*, 113(19-21):799–803.

Araujo, J., Cohen, N., de Rezende, S. F., Havet, F., and Moura, P. F. (2015). On the proper orientation number of bipartite graphs. *Theoretical Computer Science*, 566(0):59–75.

Araujo, J., Havet, F., Sales, C. L., and Silva, A. (2016). Proper orientation of cacti. *Theoretical Computer Science*, 639:14–25.

Bar-Yehuda, R., Kutiel, G., and Rawitz, D. (2016). 1.5-approximation algorithm for the 2-convex recoloring problem. In *Proceedings of the 26th International Workshop, IWOCA 2015, Verona, Italy, October 5-7, 2015, Revised Selected Papers*, pages 299–311. Springer International Publishing.

Campêlo, M., Moura, P. F., and Santos, M. C. (2013). On the representatives k-fold coloring polytope. *Electronic Notes in Discrete Mathematics*, 44:239–244. Proceedings of LAGOS'13 – VII Latin-American Algorithms, Graphs and Optimization Symposium.

Campêlo, M., Moura, P. F., and Santos, M. C. (2016). Lifted, projected and subgraph-induced inequalities for the representatives k-fold coloring polytope. *Discrete Optimization*, 21:131–156.

Campêlo, M. B., Huiban, C. G., Sampaio, R. M., and Wakabayashi, Y. (2014). Hardness and inapproximability of convex recoloring problems. *Theor. Comp. Sci.*, 533:15–25.

Kanj, I. A. and Kratsch, D. (2009). Convex recoloring revisited: Complexity and exact algorithms. In *Proceedings of the 15th Annual International Conference on Computing and Combinatorics*, COCOON '09, pages 388–397.

Knox, F., González Hermosillo de la Maza, S., Mohar, B., and Linhares Sales, C. (2016). Proper Orientations of Planar Bipartite Graphs. *ArXiv e-prints*.

Mader, W. (1967). Homomorphieeigenschaften und mittlere Kantendichte von Graphen. *Mathematische Annalen*, 174:265–268.

Mader, W. (1985). Degree and local connectivity in digraphs. *Combinat.*, 5(2):161–165.

Moura, P. F. and Wakabayashi, Y. (2017). Strong intractability of generalized convex recoloring problems. *Electronic Notes in Discrete Mathematics*, 62:93 – 98. LAGOS'17 - IX Latin and American Algorithms, Graphs and Optimization.

Moura, P. F. S. (2017). *Graph colorings and digraph subdivisions*. PhD thesis, Instituto de Matemática e Estatística, Universidade de São Paulo. http://www.teses.usp.br/teses/disponiveis/45/45134/tde-23052017-100619/.