Partition and Extension of Chordal Graphs into Independent Sets and Cliques

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Abstract. *We study the (k, l)-graphs, graphs whose set of vertices can be partitioned into k independent sets and l cliques. Recognizing (k, l)-graphs is an NP-complete problem when $k \geq 3$ or $l \geq 3$. We polynomially recognize and characterize chordal (k, l)-graphs. Moreover, we consider general M-partitions for the class of chordal graphs. For each symmetric matrix M over $0, 1, *$, the M-partition problem seeks a partition of the set of vertices into independent sets, cliques, or arbitrary sets, with pairs of sets being required to have no edges, or to have all edges joining them, as enconded in the matrix M. We show that many partition problems become polynomial time solvable for chordal graphs even in presence of lists.*

Resumo. *Estudamos os grafos-(k, l), grafos cujo conjunto de vértices pode ser particionado em k conjuntos independentes e l cliques. O reconhecimento de grafos-(k, l) é NP-completo para $k \geq 3$ ou $l \geq 3$. Caracterizamos e reconhecemos polinomialmente os grafos cordais-(k, l). Além disso, consideramos M-partições gerais para esta mesma classe. Para cada matriz simétrica M definida sobre $0, 1, *$, o problema da M-partição procura por uma partição dos vértices do grafo em conjuntos independentes, cliques, ou conjuntos arbitrários, exigindo-se todas as arestas (ou nenhuma) entre pares de conjuntos, tal como codificado na matriz M. Mostramos que muitos desses problemas de partição tornam-se polinomiais para grafos cordais mesmo na presença de listas.*

1. Introdução

1.1. Motivation

This Ph.D. thesis is a study on graph partitions. Such subject has been extensively studied in the context of graph perfection and in the search for efficient algorithms to recognize certain classes of graphs [Golumbic, 1980].

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Many combinatorial problems can be described as finding a partition of the vertices of a given graph into subsets satisfying certain properties, either *internally* (some parts may be required to be independent or sparse in some sense, others may conversely be required to be complete or dense) or *externally* (some pairs of parts may be required to be completely nonadjacent, others completely adjacent). As an example, we can consider one of the most famous problems in graph theory, the coloring problem, where our objetive is to partition the vertices of a graph into k independent sets $V_1, V_2, \ldots, V_k$ (without external restrictions). We know that this problem is polynomial time solvable for $k \leq 2$ and NP-complete for $k \geq 3$.

Another well known problem in graph partitions consists of verifying if a given graph is split, or equivalently, if its set of vertices can be partitioned into two subsets, where one of them is an independent set and the other is a clique. It has been proved that split graphs can be recognized in linear time, see for example [Golumbic, 1980].

A generalization of split graphs has been proposed by Brandstädt [Brandstädt, 1996], which introduced a new class of graphs, the (k, l)-*graphs*, consisting of those graphs for which the set of vertices can be partitioned into k independent sets and l cliques. Brandstädt proved that the (k, l)-graph recognition problem is NP-complete for $k \geq 3$ or $l \geq 3$. As a consequence of this fact, we restricted our efforts on recognizing (k, l)-graphs to a special class of graphs: the chordal graphs. A graph is said *chordal* if every induced cycle of size at least 4 has a chord, which is an edge joining two non-consecutive vertices in the cycle. We present a characterization by forbidden subgraphs for the class of chordal (k, l)-graphs, as well as a polynomial time algorithm with time complexity $O(n(n + m))$ to recognize it. In particular, we obtain a new simple and efficient greedy algorithm for the recognition of split graphs, from which it is easy to derive the well known forbidden subgraph characterization of split graphs.

These results have been presented in the International Conference *Brazilian Symposium on Graphs, Algorithms and Mathematics* [Hell et al., 2001]; and it has been accepted for publication in the journal *Discrete Applied Mathematics* [Hell et al., 2004].

A special case for chordal (k, l)-graphs was first considered in [Nogueira, 1999].

In [Hell et al., 2002b] we have presented an alternative algorithm to recognize chordal (k, l)-graphs. We formulate the chordal (k, l)-graph problem in terms of linear programming, providing two linear programs, one dual of each other. These results have been presented in the *XI Congreso Latino Iberoamericano de Investigación de Operaciones,* CLAIO 2002.

We also consider the following generalization of the (k, l)-graphs, the M-*partition* problem, as follows: partition the vertices of an input graph into k parts $V_1, V_2, \ldots, V_k$ with a fixed pattern of requirements in such a way that the A_i´s are stable or complete, and the pairs V_i, V_j are completely non-adjacent or completely adjacent. These requirements may be conveniently captured by a symmetric k-by-k matrix M in which the diagonal entries $M_{i,i}$ encode the internal restrictions on the sets V_i, and the off-diagonal entries $M_{i,j}$, $i \neq j$, encode the restrictions on the edges between V_i and V_j.

The M-partition problem was introduced in [Feder et al., 1999b]. It is easy to see that M-partitions generalize (k, l)-graphs.

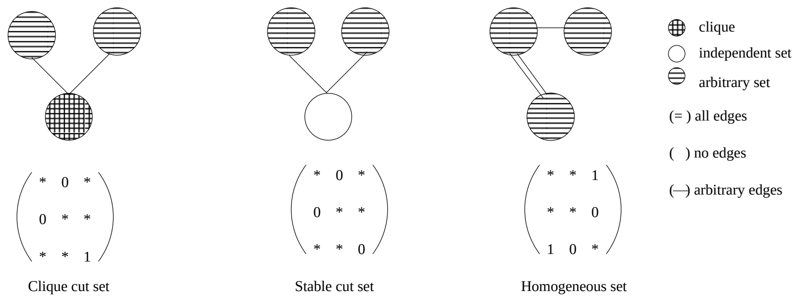

Figure 1: Some well-known M-partition problems

Another problem concerning graph partitions is the *list* M*-partition* problem [Feder et al., 2003]. An instance of the list M-partition problem is a graph G, together with a collection of *lists* $L(x)$, $x \in V(G)$, each list being a set of parts. A solution for the instance of list M-partition is a solution for the corresponding M-partition, such that each vertex x is placed in a part $i \in L(x)$.

List partititons generalize list colorings and list homomorphisms [Feder et al., 1999a]. Each symmetric matrix M over $0, 1, *$ defines a list partition problem. Different choices of the matrix M lead to many well-known graph theoretic problems including the problem of recognizing split graphs and their generalizations, finding homogeneous sets, joins, clique cutsets, stable cutsets, skew cutsets and so on. (Figure 1 illustrates some examples.)

In this thesis [Nogueira, 2003], we considered the restricitions of both the M-partition and the list M-partition problems to instances G that are chordal graphs (besides the special problem of recognizing chordal (k, l)-graphs). The two corresponding problems will be called the *chordal* M*-partition problem* and *chordal list* M*-partition problem*.

Results concerning chordal M-partitions and chordal list M-partitions have been accepted for presentation in the conference *Latin American Theoretical INformatics*, LATIN 2004 and accepted for publication in the LNCS series [Feder et al., 2004].

2. Characterizing Chordal (k, l)-graphs

As an important result of this thesis, we prove that in a chordal graph the maximum number of independent (i.e. disjoint and non-adjacent) K_r's equals the minimum number of cliques that meet all K_r's [Hell et al., 2002a], [Hell et al., 2002b]. When $r = 1$, this implies that chordal graphs are perfect. When $r = 2$, it contains a well known forbidden subgraph characterization of split graphs. We also discuss algorithms for both cases, generalizing these results to chordal (k, l)-graphs. Much of the appeal of split graphs is due to the fact that they are chordal, a property not shared by (k, l)-graphs in general. (For instance, being a $(k, 0)$-graph is equivalent to being k-colourable.) However, if we keep the assumption of chordality, nice algorithms and characterizations theorems are possible. Indeed, our result gives a forbidden subgraph characterization of (and a polynomial time recognition algorithm for) chordal (k, l)-graphs.

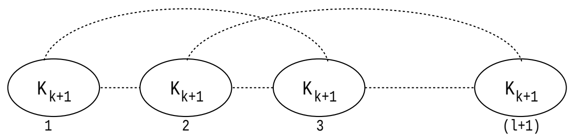

Figure 2: $(l+1)K_{k+1}$

2.1. The Theorems

In [Brandstädt, 1998], $O((n+m)^2)$ recognition algorithms for $(2,1)$-, $(1,2)$-, and $(2,2)$-graphs, are given. In what follows we present our forbidden subgraph characterization of chordal (k,l)-graphs and briefly discuss a polynomial time recognition algorithm for this class of graphs.

Let r be a positive integer. Given a graph G, we denote by $\alpha(G,r)$ the maximum number of independent copies of K_r's in G, and by $\kappa(G,r)$ the minimum number of cliques of G which meet all K_r's of G. We prove that the equality $\alpha(G,r) = \kappa(G,r)$ holds for chordal graphs. Observe that this equality holds for perfect graphs when $r = 1$, since it corresponds exactly to the equality between independence number and clique covering number.

This result depends on some facts about the structure of cliques in chordal graphs. We have the following observations:

Observation 1 *If C and K are two disjoint cliques in a chordal graph G, then there exists a clique C' with the following property: C' intersects K, and it also intersects all the cliques adjacent to K which are intersected by C.*

Observation 2 *For any collection $C_1, \ldots, C_p$ of pairwise adjacent cliques in a chordal graph G, there exists a clique C in G which intersects each C_i, $i = 1, \ldots, p$.*

A simple necessary condition for a graph G to be a (k,l)-graph is that it does not contain $l+1$ independent K_{k+1}'s (Figure 2). It turns out that for chordal graphs the above condition is also sufficient. We use the following theorem to prove this fact.

Theorem 1 *Let G be a chordal graph, and $r \geq 1$ be an integer. Then $\alpha(G,r) = \kappa(G,r)$.*

Sketch of the Proof: It is clear that $\alpha(G,r) \leq \kappa(G,r)$. For the converse, we introduce the following construction: Define the graph $K^r(G)$, which has a vertex for each copy of a K_r in G, and two vertices of $K^r(G)$ are adjacent if and only if the corresponding K_r's are adjacent (i.e. not independent) in G. It is clear that for any graph G, $\alpha(G,r)$ is the independence number of $K^r(G)$. Moreover, for a chordal graph G, $\kappa(G,r)$ is the clique covering number of $K^r(G)$: Indeed, we can modify any clique cover of $K_r(G)$ with s cliques to construct s cliques of G which meet all K_r's of G, by applying the previous observations to each collection $C_1, \ldots, C_s$ of cliques of G corresponding to a clique of the clique cover of $K^r(G)$. It can be shown that if G is chordal then $K^r(G)$ is also chordal. In particular, $K^r(G)$ is perfect. Thus the independence number of $K^r(G)$ is equal to the clique covering number of $K^r(G)$, and hence to $\kappa(G,r)$ as noted above.

In the following, we state one of the main result of this thesis.

Theorem 2 *A chordal graph is a (k, l)-graph if and only if it does not contain $l + 1$ independent K_{k+1}'s.*

Sketch of the Proof: We know that any (k, l)-graph cannot contain $l + 1$ independent K_{k+1}'s. On the other hand, Theorem 1 implies that if a chordal graph G does not contain $l + 1$ independent copies of K_{k+1}, then $\kappa(G, k + 1) \leq l$. This means that G contains l cliques whose removal leaves a subgraph G' without K_{k+1}. Since G is perfect, G' is k-colourable, whence G admits a partition into k inpendent sets and l cliques.

2.2. The Algorithms

When k and l are fixed, Theorem 2 implies a polynomial time recognition algorithm for chordal (k, l)-graphs. There are, however, more efficient algorithms. We present an $O(n(n + m))$ time algorithm to recognize chordal (k, l)-graphs [Nogueira, 2003], [Hell et al., 2002c], [Hell et al., 2004]. In fact, this algorithm finds a minimum value l such that G is a (k, l)-graph. The algorithm is more efficient when $k = 1$, i.e., when we seek a partition into *one* independent set and a set of cliques. When both k and l are one, we specialize the algorithm to yield a new simple and efficient recognition algorithm for split graphs. The value of the algorithm is underscored by the fact that it easily adapts to solve the list version of the split partition problem - finding an extension of a given pre-assignment of some vertices to the independent set, or clique. As a byproduct of the algorithm we also obtain a forbidden subgraph characterization of when such an extension is possible.

Another algorithm we have produced is described in terms of $\alpha(G, r)$ and $\kappa(g, r)$, where $r \leq 1$ is a fixed integer. (Recall from above, that a chordal graph G is a (k, l)-graph if and only if $\kappa(G, k + 1) < l$, then a partition of G into k independent sets and l cliques is found by the standard greedy algorithm for k-colouring G.

We give two linear programs P and D [Hell et al., 2002b], dual to each other. Each integer solution to P corresponds to a set of cliques of G which meet all the K_r's and conversely. Thus P computes the fractional relaxation of $\kappa(G, r)$. Each integer solution of D corresponds to a set of independent K_r's and conversely. Thus D computes the fractional relaxation of $\alpha(G, r)$. We describe a greedy algorithm to obtain a particular integer solution of D, and another algorithm which uses this greedy solution to obtain a particular integer solution of P. It will turn out (by the complementary slackness principle) that these two solutions are both optima, thus yielding $\alpha(G, r)$, $\kappa(G, r)$, as well as a second proof of Theorem 1. With these results, we also solve the weighted version: Given a graph G with a nonnegative integer weight for each K_r in G, we seek a set of independent K_r´s with maximum total weight.

3. M-Partitions

As another important contribution of this thesis, we expand our focus and consider M-partitions for the class of chordal graphs. The M-partition problem was introduced in [Feder et al., 1999b]. Let M be a symmetric $m \times m$ matrix with entries $M_{i,j} \in \{0, 1, *\}$. An instance of the M-partition problem is a grah G. A solution for the the instance is a partition of vertices in G into m *parts*, corresponding to the rows (and columns) of the matrix M, such that for distinct vertices x and y of the graph G, placed in parts i and j (possibly $i = j$) respectively, we have the following:

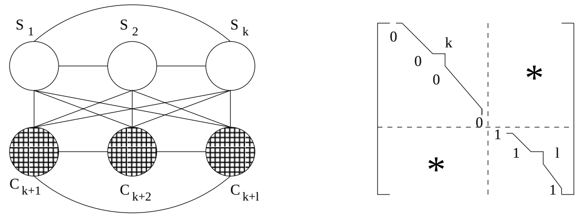

Figure 3: A (k, l)-graph and its coresponding matrix

- if $M(i, j) = 0$, then xy is not an edge of G;
- if $M(i, j) = 1$, then xy is an edge of G.

(If $M(i, j) = *$, then xy may or may not be an edge in G.)

Figure 3 depicts a (k, l)-graph and its corresponding matrix, suggesting that M-partition generalizes the (k, l)-graphs.

There are several classical examples to suggest that M-partitions of cordal graphs can be found in polynomial time. For instance, k-colorability of chordal graphs (M is the $k \times k$ matrix with 0 on the diagonal and $*$ everywhere else) can be decided efficiently using a perfect elimination ordering [Golumbic, 1980]; in fact, the algorithm produces a k-coloring of the input graph or produces the unique forbidden subgraph K_{k+1}. A similar result is known about clique covering (M is the $l \times l$ matrix with 1 on the diagonal and $*$ everywhere else). In [Hell et al., 2004] we have shown more generally that there is a polynomial time recognition algorithm, and a forbidden subgraph characterization of graphs that can be partitioned into k independent sets and l cliques (M has k 0's and l 1's on the diagonal, $*$ everywhere else). We extend these results to the list M-partition problem. We also extend the class of matrices M for which we can give polynomial time algorithms, and forbidden subgraph characterizations.

4. Matrices M with $0, 1$ diagonal

If the diagonal of the matrix M contains no $*$, we have several large classes of polynomial time solvable list M-partition problems, including the list versions of the above problem of partitioning G into k independent sets and l cliques. We have the following theorems:

Theorem 3 *If all diagonal entries of M are 0, then the chordal list M-partition problem can be solved in polynomial time.*

One can observe that the case above solves the usual k-colorability problem with list for chordal graphs. In [Hell et al., 2003], we specialize this result by showing that when the diagonal entries of M are 0 and it has asterisks everywhere else then the list M-partition problem for chordal graphs can be solved in linear time. Moreover, we provide a linear time algorithm when the offdiagonal entries of M are $*$´s and the diagonal entries has one 1 and the rest of the diagonal entries are 0.

For the case where the $k \times k$ matrix M has only 1's in the diagonal, we have the following result:

Theorem 4 *If all diagonal entries of M are 1, then the chordal list M-partition problem can be solved in time polynomial to n^l.*

Other results consider $(k, l) \times (k, l)$ matrices M which consist of a $k \times k$ diagonal matrix A and an $l \times l$ matrix B, with an off-diagonal $k \times l$ matrix C (and its $l \times k$ transpose). We call such matrices A, B, C*-block matrices*. Assume now that all diagonal entries of A are 0, and all diagonal entries of B are 1. We shall consider restrictions on C. For this, we are going to define some special kind of matrices.

We call a matrix C *horizontal* if all entries of C corresponding to a part i in A are the same, and *vertical* if all entries of C corresponding to a part j in B are the same. Finally, we call matrix C *crossed* if the entries of C are all 0 or $*$ (or all 1 or $*$) and every 0 (respectively 1) belongs to either a row or a column of all 0's (respectively all 1's).

Theorem 5 *Suppose M is an A, B, C-block matrix. If all diagonal entries of A are 0, all diagonal entries of B are 1, and if C is either horizontal, vertical, or crossed, then the chordal list M-partition problem can be solved in polynomial time in $O(n^{kl})$.*

4.1. NP-complete results

Consider a fixed bipartite graph H. The *list H-coloring problem* is defined as follows: An instance is a bipartite graph G with lists (white vertices of G have lists consisting of white vertices of H and similarly for black vertices), and a solution is a mapping of vertices of G to vertices of H so that adjacency is preserved and each vertex of G is mapped to a member of its list. (Such a mapping is called a *list H-coloring* of G.) [Feder et al., 1999a] showed that the list H-coloring problem is polynomial time solvable if the bipartite graph H is the complement of a circular arc graph (a cocircular graphs), and is NP-complete otherwise. Based on this result, it will be possible to find NP-complete chordal list M-partition problems.

Given a bipartite graph H with k white vertices (forming the set V_A) and l black vertices (forming the set V_B), the matrix corresponding to H is the $k \times l$ matrix C with $C(i, j) = *$ if ij is an edge in H (with $i \in V_A$, $j \in V_B$), and with $C(i, j) = 0$ otherwise.

Theorem 6 *Let M be an A, B, C-block matrix. Suppose A does not contain any 1's, and B does not contain any 0's. If C is the matrix corresponding to a bipartite graph H that is not a cocircular graph, then the chordal list M-partition problem is NP-complete.*

The above theorem implies that the list M-partition problem corresponding to graphs that are not cocircular are NP-complete even for split graphs.

5. Conclusion

We have shown new results for some graph partition problems. We have presented a characterization for chordal (k, l)-graphs by forbidden subgraphs. Moreover, we have presented a polynomial time algorithm for recognizing this class of graphs. We have also presented a new algorithm to recognize split graphs providing a characterization by forbidden subgraphs for the cases with or without lists.

For the M-partition problem with lists, we have shown that there exist matrices M for which the problem is polynomial time solvable and we have also shown that there exist others for which the problem is NP-complete (even for the case whithout lists).

As a byproduct of this thesis, we have shown [Feder et al., 2004] that there are M-partition problems that remain NP-complete even for chordal graphs (without lists). We also discuss forbidden characterizations for the existence of M-partitions.

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