Novel Procedures for Graph Edge-colouring

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Abstract. *We present a novel recolouring procedure for graph edge-colouring. We show that all graphs whose vertices have local degree sum not* too *large can be optimally edge-coloured in polynomial time. We also show that the set of the graphs satisfying this condition includes* almost every *graph (under the uniform distribution). We present further results on edge-colouring join graphs, chordal graphs, circular-arc graphs, and complementary prisms, whose proofs yield polynomial-time algorithms. Our results contribute towards settling the* Overfull Conjecture*, the main open conjecture on edge-colouring simple graphs. Finally, we also present some results on total colouring.*

Resumo. *Apresentamos um procedimento novo de recoloração para coloração de arestas de grafos. Mostramos que todos os grafos cujos vértices têm soma local de graus não alta* demais *podem ter suas arestas coloridas em tempo polinomial. Também mostramos que* quase todo *grafo (na distribuição uniforme) satisfaz essa condição. Ainda exibimos resultados em grafos-junção, grafos cordais, grafos arco-circulares, e prismas complementares, cujas provas levam a algoritmos polinomiais. Nossos resultados contribuem na direção de resolver a* Conjectura Overfull*, a principal conjectura em coloração de arestas de grafos simples. Por fim, apresentamos resultados em coloração total.*

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1. Introduction and motivation

Computing an optimal edge-colouring of a graph is an NP-hard [Holyer 1981] combinatorial problem with applications in network protocols [Erlebach and Jansen 2001, Gandham et al. 2005] and task scheduling in industry [Williamson et al. 1997]. There has been much work aimed at identifying sets of instances for which the problem becomes polynomial (e.g. [Chetwynd and Hilton 1989, Bodlaender 1990, Ortiz Z. et al. 1998]) and sets of instances for which NP-hardness remains (e.g. [Cai and Ellis 1991, Koreas 1997, Machado et al. 2010]). We refer the reader to the thesis, Sect. 1.6, p. 26, for an extensive table on the complexity of edge-colouring restricted to several graph classes.

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By Vizing's Theorem [Vizing 1964], the chromatic index of any simple graph G (i.e. the least k for which G is k-edge-colourable) is either its maximum degree Δ or $\Delta+1$. The graph G is said to be *Class 1* in the former case, or *Class 2* in the latter. The *core* (i.e. the subgraph induced by the vertices of maximum degree) of *almost every*[1] graph is unitary [Erdős and Wilson 1977], hence acyclic, and graphs with acyclic core are *Class 1* and admit a polynomial-time exact edge-colouring algorithm [Fournier 1977], implying that the set of instances for which edge-colouring is hard includes *almost no* graph.

Intriguingly, edge-colouring is NP-hard even for graph classes such as perfect graphs [Cai and Ellis 1991]. For many other classes, even well-structured ones such as cographs or chordal graphs, the complexity of the problem remains open, with apparently few progress on long-standing conjectures despite much work (see Chap. 2 of the thesis).

All graphs considered in this text are simple. The set of neighbours of a vertex u in a graph G is denoted $N_G(u)$. A *major* of G is a vertex of maximum degree in G, and the *local degree sum* of a vertex is the sum of the degrees of its neighbours. The *core* of G is denoted $\Lambda[G]$. Other graph-theoretical concepts follow their usual definitions.

An n-vertex graph with maximum degree Δ is said to be *overfull* if it has more than $\Delta\lfloor n/2\rfloor$ edges. A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an *overfull* subgraph H with $\Delta(H)=\Delta(G)$. Clearly, every *SO* graph G is *Class 2*, but examples of non-*SO Class 2* graphs are known (e.g. the Petersen graph). The *Overfull Conjecture* states that being *SO* is equivalent to being *Class 2* for n-vertex graphs with $\Delta > n/3$ [Chetwynd and Hilton 1984, Chetwynd and Hilton 1986, Hilton and Johnson 1987]. Deciding if a graph is *SO* can be done in polynomial time [Padberg and Rao 1982].

2. Main results and impact

In this section we present some of the results achieved and briefly discuss their impact.

Theorem 1 (Theorem 1.10 in the thesis). *Let $\mathscr{X}$ be the class of the graphs with maximum degree Δ whose majors have local degree sum bounded above by $\Delta^2-\Delta$. All graphs in $\mathscr{X}$ are Class 1.*

We also show that *almost every* graph is in $\mathscr{X}$, even given that the graph has cycles in the core, that is, even when restricted to the instances which have not been ruled out by the aforementioned results of [Erdős and Wilson 1977] and [Fournier 1977].

The proof of Theorem 1 lies on a novel recolouring procedure with which an optimal edge-colouring can be constructed in polynomial time edge by edge for any graph in $\mathscr{X}$ (see Sect. 3 of this text and Chap. 3 of the thesis). Further, the development of this recolouring procedure contributes towards settling the Overfull Conjecture. As discussed in Sect. 6.1 of the thesis, [Niessen 1994] suggests that a step towards a proof for the Overfull Conjecture is to prove the following.

Conjecture 2. *If u is a vertex of a graph G adjacent to at most one major of G with local degree sum at least $\Delta^2-\Delta+2$, and if $\chi'(G-u)\leq\Delta$, then G is Class 1.*

Using our recolouring procedure we have proved a slightly weaker form of this conjecture, just replacing the lower bound $\Delta^2-\Delta+2$ by $\Delta^2-\Delta+1$ (see the proof for

[1]When we say that *almost every* graph satisfies a property $\mathscr{P}$, we mean that the probability of a uniformly sampled ($\mathscr{G}_{n,1/2}$) n-vertex graph to satisfy $\mathscr{P}$ goes to 1 as n goes to ∞.

Theorem 3.17 in the thesis). This and the fact that almost every graph is in $\mathscr{X}$ enhance the importance of our recolouring procedure.

Below we state our main result on complementary prisms.

Theorem 3 (Theorem 1.13 in the thesis)**.** *A complementary prism can be Class 2 only if it is a regular graph distinct from the K_2.*

We have also proved that no complementary prism can be *SO* (Theorem 4.30 in the thesis). This implies that a *Class 2* (regular) complementary prism would serve as an example of our generalisation of the definition of the snarks for d-regular graphs for $d \geq 3$ odd (see Sect. 4.4 of the thesis). Recall that snarks constitute an important graph class much related to the history of the *Four Colour Theorem* (see Chap. 2 of the thesis).

The result below is a special case of a conjecture on edge-colouring chordal graphs, open for more than two decades, which states that all chordal graphs of odd maximum degree are *Class 1* [Figueiredo et al. 1995].

Theorem 4 (Theorem 4.36 in the thesis)**.** *Except for the K_3, all chordal graphs with maximum degree $\Delta \leq 3$ are Class 1.*

Theorem 5 (i) and (ii) below contribute towards settling the Overfull Conjecture for join graphs and cographs (see Chap. 2 and 4 of the thesis), since in these theorems we prove that some graphs which we know that are not *SO* are in fact *Class 1*.

Theorem 5. *Let $G := G_1 * G_2$ be a join graph with $n_1 \leq n_2$ and $\Delta_1 \geq \Delta_2$ (we define $n_i := |V(G_i)|$ and $\Delta_i := \Delta(G_i)$ for $i \in \{1, 2\}$). The following are sufficient conditions for G to be Class 1:*

(i) (Theorem 4.9 in the thesis) $\Lambda[G_1]$ *is acyclic and, being $T_1, \ldots, T_k$ the connected components of* $\Lambda[G_1]$,

$$|\{u \in V(\Lambda[G_1]) : d_{\Lambda[G_1]}(u) > 1\}| + |\{T_i : |V(T_i)| = 2\}| \leq n_2 - |V(\Lambda[G_2])|\,;$$

(ii) (Theorem 4.18 in the thesis) $n_2 - n_1 \geq 2$.

We remark that the proofs for all the aforementioned results are constructive and yield polynomial-time edge-colouring algorithms. Chap. 5 of the thesis also presents some results on total colouring.

3. The recolouring procedure

We present a sketch of our recolouring procedure (see Chap. 3 of the thesis for the details). Throughout this section, let $\varphi\colon E(G) \setminus \{uv\} \to \mathscr{C}$ be a $\Delta(G)$-edge-colouring of $G - uv$ for some edge $uv \in E(G)$, which is the edge about to be coloured by the procedure.

If the vertex u and all its neighbours miss at least one colour of $\mathscr{C}$ each (we say that a vertex x *miss* a colour α if no edge incident to x is coloured α), then we can use Vizing's recolouring procedure in order to obtain a colour of $\mathscr{C}$ to assign to uv without creating colour conflicts [Vizing 1964]. However, since $|\mathscr{C}| = \Delta(G)$, the neighbours of u in $G - uv$ considered by the procedure may not *actually* miss at least one colour of $\mathscr{C}$, but can still be handled as long as they *virtually* miss at least one colour of $\mathscr{C}$, as defined below. Our procedure extends Vizing's in the sense that if every considered vertex actually misses a colour, then our procedure behaves exactly as Vizing's.

Definition 6. A sequence $v_0, \dots, v_k$ of *distinct* neighbours of u in G is a *recolouring fan* for uv if $v_0 = v$ and, for all $i \in \{0, \dots, k-1\}$: either v_i actually misses the colour $\alpha_i := \varphi(uv_{i+1})$; or v_i misses the colour $\alpha_i := \varphi(uv_{i+1})$ *virtually*, that is, $i > 0$ and $\varphi(v_i w_i) = \alpha_i$ for some $w_i \in N_G(v_i) \setminus \{v_{i-1}\}$ which actually misses α_{i-1}.

We show that if there is a recolouring fan $v_0, \dots, v_k$ for uv such that v_k misses (actually or virtually) a colour which is

(Condition 1) either missing at u,
(Condition 2) or missing (actually or virtually) at v_j for some $j < k$,

then the edges of $G - uv$ can be recoloured in order to obtain a colour to be assigned to uv without creating colour conflicts. Although the proof under Condition 1 goes quite straightforwardly, as Fig. 1 illustrates, several challenging complications rise in order to guarantee the proof under Condition 2, which we have successfully managed (see Lemmas 3.1–3.7, p. 51–59, in the thesis). Fig. 2 illustrates one of these complications.

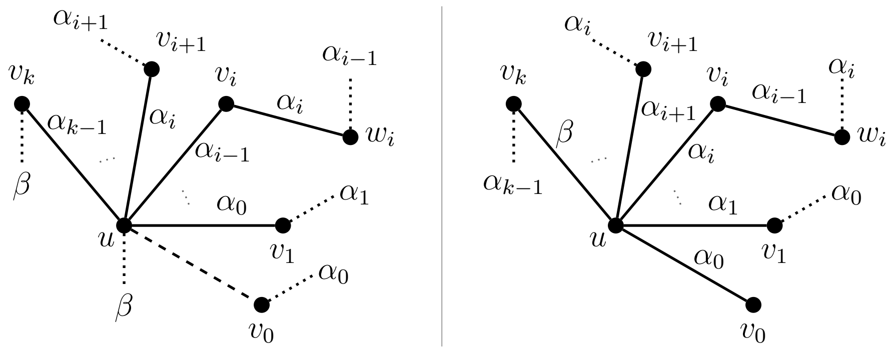

Figure 1. A recolouring fan $v_0, \dots, v_k$ for uv such that v_k actually misses a colour β which is missing at u. The dotted lines indicate the colours actually missing at the vertices, and the dashed line the edge to be coloured.

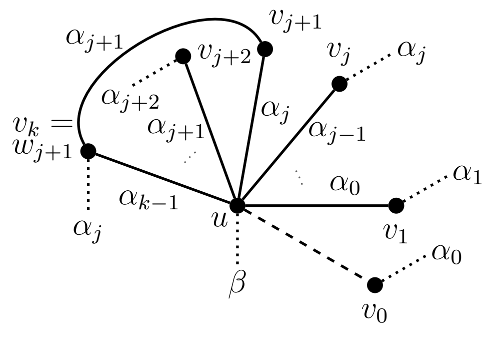

Figure 2. A recolouring fan $v_0, \dots, v_k$ for uv such that v_k misses α_j for some $j < k$, but the vertex v_{j+1} misses α_{j+1} virtually and $v_k = w_{j+1}$.

We show that if all neighbours of u in G have local degree sum bounded above by $\Delta^2 - \Delta$, then we can always recolour the edges of $G - uv$ in order to construct a recolouring fan for uv satisfying Condition 1 or Condition 2 (Lemma 3.8, p. 59–60, in the thesis). This yields the constructive proof of Theorem 1 (in the thesis, Theorem 1.10).

The proof that almost every graph is in $\mathscr{X}$ follows from the fact that almost every graph G has a single vertex u of degree at least $\frac{1}{2}\left((n-1)+(1-\varepsilon)n\ln n\right)^{\frac{1}{2}}$ for any $\varepsilon > 0$ [Erdős and Wilson 1977], but no vertex of degree exactly $d_G(u) - 1$, as we show (see Theorem 3.9 in the thesis, p. 61–62, for details).

4. Conclusion

We have presented a novel recolouring procedure for edge-colouring, with which we have achieved the result stated in Theorem 1 and a slightly weaker form of Conjecture 2, contributing towards settling the Overfull Conjecture. We have also shown that almost every graph is covered by Theorem 1 and thus can be edge-coloured in polynomial time.

We have also achieved results on edge-colouring graph classes such as complementary prisms, chordal graphs, and join graphs, contributing towards settling the Overfull Conjecture for those classes (see Chap. 4 of the thesis for more results achieved).

Finally, we suspect the following stronger form of Theorem 5(i) (Theorem 4.9 in the thesis), for which we also develop a recolouring procedure (see Sect. 4.3 of the thesis).

Conjecture 7 (Conjecture 1.11 in the thesis)**.** *Let $G := G_1 * G_2$ be a join graph with $n_1 \leq n_2$ and $\Delta_1 \geq \Delta_2$. If $\Lambda[G_1]$ is acyclic, then G is Class 1.*

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Appendix: List of publications

1. Zorzi, A. and Zatesko, L. M. (2018). On the chromatic index of join graphs and triangle-free graphs with large maximum degree. *Discrete Appl. Math.* (JCR 0.932), 245:183–189.
2. Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2018). Upper bounds for the total chromatic number of join graphs and cobipartite graphs. In *Proc. ICORES '18*, p. 247–253, Funchal, Portugal.
3. Bernardi, J. P. W., Silva, M. V. G., Guedes, A. L. P., and Zatesko, L. M. (2019). The chromatic index of proper circular arc graphs of odd maximum degree which are chordal. Accepted for presentation at *LAGOS '19*, Belo Horizonte.
4. Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2017). Edge-colouring of triangle-free graphs with no proper majors. In *Proc. CSBC '17/IIETC*, p. 71–74.
5. Bernardi, J. P. W., Almeida, S. M., and Zatesko, L. M. (2018). On total and edge-colouring of proper circular-arc graphs. In *Proc. CSBC '18/IIIETC*, p. 73–76.
6. Bernardi, J. P. W., Almeida, S. M., and Zatesko, L. M. (2018). A decomposition for edge-colouring. In *Proc. LAWCG '18*, page 35, Rio de Janeiro.
7. Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2018). A recolouring procedure for total colouring. In *Proc. LAWCG '18*, page 31, Rio de Janeiro.
8. Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2017). On a conjecture on edge-colouring join graphs. In *Proc. WPCCG '17*, p. 69–72, Ponta Grossa.

- Zatesko, L. M., Zorzi, A., Carmo, R., and Guedes, A. L. P. (2018). Edge-colouring graphs with bounded local degree sums. Submitted to *Discrete Appl. Math.*
- Zatesko, L. M., Bernardi, J. P. W., Almeida, S. M., Carmo, R., and Guedes, A. L. P. (2019). A connectivity-based decomposition for graph edge-colouring. Submitted to *Matemática Contemporânea*.