A Study of Critical Snarks*†

Breno Lima de Freitas[1], **Cândida Nunes da Silva**[1], **Cláudio L. Lucchesi**[2]

[1] DComp – CCGT – Federal University of São Carlos (UFSCar)
18052-780 – Sorocaba – SP – Brazil

[2]Faculty of Computing, FACOM-UFMS
PO Box 549 – 79070-900 – Campo Grande – MS – Brazil

brenolimadefreitas@gmail.com, candida@ufscar.br, lucchesi@facom.ufms.br

Abstract. *Snarks are cubic graphs that do not admit a 3-edge-colouring and that are regarded to be the minimal cubic graphs without this property. Snarks have been studied by many researchers throughout the history, since many famous open problems are known to have their potential counter-examples residing in this family of graphs. In this paper we present relations between several classes of critical snarks. It follows from one of such relations that no hypohamiltonian snark is a counter-example to Tutte's 5-flow Conjecture, thus giving a positive answer to a question proposed by Cavicchioli et al. in 2003.*

Resumo. *Snarks são grafos cúbicos que não admitem 3-coloração de arestas e que são considerados os grafos cúbicos minimais sem esta propriedade. Snarks vêm sendo estudados por vários pesquisadores no decorrer da história, uma vez que é sabido que vários problemas abertos famosos devem ter seus potenciais contra-exemplos dentro desta família de grafos. Neste artigo apresentamos relações entre várias classes de snarks críticos. É consequência de uma destas relações que nenhum snark hipohamiltoniano é um contra-exemplo para a Conjectura dos 5-fluxos de Tutte. Essa asserção responde afirmativamente a uma questão proposta por Cavicchioli et al. em 2003.*

1. Introduction

A *k-edge-colouring* is an assignment of at most k colours to the edges of a graph such that no two adjacent edges are assigned the same colour. The *chromatic index* of a graph G is the minimum k such that G has a k-edge-colouring. It is well known that cubic graphs fall into two categories regarding their chromatic index. By Vizing's Theorem (see [Bondy and Murty 1976][Theorem 6.2]) a cubic graph has chromatic index either 3 or 4. A *snark* is, in essence, a cubic graph that does not admit a 3-edge-colouring. Snarks are usually also required to have girth at least five and be cyclically 4-edge-connected, in order to avoid triviality. The importance of the study of snarks comes from the fact that for many famous open problems in Graph

*Support by FAPESP, CNPq and CAPES

†Most of the results presented here appear in a paper entitled "Hypohamiltonian Snarks Have a 5-Flow" which was accepted for publication on VIII Latin-American Algorithms, Graphs and Optimization Symposium

Theory it suffices to prove them for snarks. For instance, the Berge-Fulkerson's Conjecture [Fulkerson 1971], Tutte's 5-flow Conjecture [Tutte 1954] and the Cycle Double Cover Conjecture [Seymour 1979, Szekeres 1973]. Snarks were named by Gardner [Gardner 1976] after a creature from the poem "The Hunting of the Snark" by Lewis Carroll. In the poem, the snark is a mysterious and legendary creature. The apparent difficulty to find non-3-edge-colourable cubic graphs inspired the name. Many references, such as [Bondy and Murty 2008, Chetwynd and Wilson 1981, West 1996], present interesting accounts of the appealing history behind the search for snarks.

The study of cubic graphs dates back to the Four Colour Problem, one of the oldest and most studied problems in Graph Theory. The Four Colour Problem was one of first problems known to be reducible to the class of cubic graphs. In 1880 Tait presented a proof that the faces of a cubic planar map without a 1-edge cut could be coloured with at most four colours if and only if it had a 3-edge-colouring. So far, no snark was known. Only in 1898 the first snark was discovered by Julius Petersen; it was later named Petersen graph. The Petersen graph is also the smallest possible snark. For many of the following decades, few other snarks were discovered. However, in [Isaacs 1975] it was shown that there were infinitely many snarks. Isaacs presented the infinite family of *flower-snarks* and also described an operation, called the *dot product* which could be used to obtain a new snark out of two snarks.

Let $k > 1$ be an integer, let G be a graph, let D be an orientation of G and let φ be a weight function that associates to each edge of G a positive integer in the set $\{1, 2, \ldots, k-1\}$. The pair (D, φ) is a *(nowhere-zero) k-flow* of G if every vertex v of G is *balanced*, i. e., the sum of the weights of all edges leaving v equals the sum of the weights of all edges entering v. Tutte [Tutte 1954] defined the concept of k-flow as a generalization of the concept of k-face-colouring after observing that, for any planar graph, a k-flow can be obtained from a k-face-colouring and vice-versa (a proof can be found in [Younger 1983]). The concept of k-flow generalizes that of a k-face-colouring since it does not depend on an embedding of the graph in a particular surface. In this paper we focus our attention on Tutte's 5-flow Conjecture, stated below.

Conjecture 1.1 (Tutte's 5-Flow Conjecture [Tutte 1954]) *Every graph with no 1-edge cut admits a 5-flow.*

Several partial results can be found in the literature. For instance, Zhang [Zhang 1997], Diestel [Diestel 1996] and Seymour [Seymour 1995] bring many partial results regarding Tutte's Conjectures. The most important of all is Seymour's 6-Flow Theorem [Seymour 1981], which states that every graph with no 1-edge cut admits a 6-flow. This is the best known approximation for Tutte's 5-Flow Conjecture.

Tutte also proved the following Theorem, which is particularly relevant to this work. Its proof is simple and elegant, and can be found in [Bondy and Murty 2008, Theorem 21.11].

Theorem 1.2 (Tutte) *A cubic graph admits a 4-flow if and only if it admits a 3-edge-colouring.*

Tutte's 5-Flow Conjecture remains open. Also, no simple characterization for snarks is known. Given the amount of research done so far regarding both problems, it is clear that these are hard problems. This work aims at giving a small contribution to both of them. More specifically, we studied how several classes of critical snarks relate to each other. Among the classes studied there are snarks that are critical for being somehow close to having a 4-flow and others that are critical for being somehow close to having a 3-edge-colouring. We observed that two of such classes are equivalent and also that hypohamiltonian snarks are included in this class. As a result, a question proposed by Cavicchioli et al. in 2003, namely whether every hypohamiltonian snark has a 5-flow, could be positively answered.

2. Concepts of criticality

A graph is said to be *critical* regarding a property, if it does not have that mathematical property but is somehow close to having it. Typically, when some reduction operation is applied to the critical graph, the resulting graph has the desired property. The definition of many classes of critical graphs regarding both edge-colouring and k-flow properties can be found in the literature. In this section, we give the definition of the classes of critical graphs that were studied in this work.

Let G be a graph. We denote by G/e the graph obtained from G by the contraction of edge e. Let S be a set of edges of G, then we denote by $G \setminus S$ the graph obtained from G after the removal of all edges in S. In particular, when S is a trivial set containing only edge e we denote it by $G \setminus e$. We denote by $G \setminus v$ the graph obtained from G after the removal of vertex v and by $G \setminus \{u, v\}$ the graph obtained from G after the removal of a pair of vertices u and v.

A graph is *2-vertex-(colour)-critical* if the removal of two adjacent vertices lowers its chromatic index. A graph is *2-vertex-(colour)-co-critical* if the removal of two non-adjacent vertices lowers its chromatic index. A graph is *(colour)-bicritical* if it is 2-vertex-critical and 2-vertex-co-critical simultaneously.

A k-factor is a spanning k-regular subgraph. Therefore, a 1-factor is a perfect matching and a 2-factor is a spanning set of cycles. Every cubic graph with no 1-edge cut has a perfect matching and thus can be decomposed into a 1-factor and a 2-factor. We define the *oddness* of a graph G, denoted by $\omega(G)$, as the minimum number of odd cycles in any 2-factor of G. Jaeger was the first to research on graphs with small oddness (see [Jaeger 1988]).

Proposition 2.1 *A cubic graph G is 3-edge-colourable if and only if* $\omega(G) = 0$.

Proof Let G be a 3-edge-colourable cubic graph. We denote by M_i the set of edges coloured with colour i, for $i = 1, 2, 3$. Recall that $M_1 \cup M_2 \cup M_3 = E(G)$ and each M_i is a perfect matching of G. Since $G \setminus M_1$ is a 2-factor of G, it follows that every cycle has an even number of vertices.

Assume now that $\omega(G) = 0$. Let M be a perfect matching of G, such that $G \setminus M$ is a 2-factor containing even cycles only. Then, $G \setminus M$ is 2-edge-colourable. It follows that assigning a third colour to M yields a 3-edge-colouring of G. ∎

It follows from Proposition 2.1 that no snark has oddness zero. Moreover, the oddness of every cubic graph is always even, as every cubic graph has even order. In [Steffen 2004], a concept which is closely related to the oddness of a graph, called resistance, was introduced. The *resistance* of a cubic graph G, denoted by $\rho(G)$, is the minimum number of edges that must be removed in order to obtain a 3-edge-colourable graph. Note that $\rho(G) \leq \omega(G)$, because the removal of one edge of each odd cycle in a 2-factor of G yields a 3-edge-colourable graph.

Every cubic Hamiltonian graph has oddness zero and therefore has chromatic index 3. It follows that no snark is Hamiltonian. A graph G is called *hypohamiltonian* if $G \setminus v$ is Hamiltonian for every vertex v of G. A large number of famous snarks are both hypohamiltonian and bicritical. In [Steffen 1998][Theorem 4.13] it is shown that every hypohamiltonian snark is also bicritical.

In [da Silva and Lucchesi 2008] the concept of a *k-flow-critical* graph was introduced with the purpose of studying the possible definitions of a minimal graph without a k-flow and also understanding the structure of such graphs. Da Silva and Lucchesi observed several analogies between properties satisfied by k-flow-critical graphs and by the k-vertex-colour-critical graphs well studied by Dirac[Dirac 1951]. A graph G is *k-(edge)-flow-critical* if it does not admit a k-flow but the graph G/e admits a *k-flow* for every edge e of G. Clearly, as contracting an edge does not create any new cuts, every minimum counterexample to Tutte's 5-Flow Conjecture is a k-flow-critical graph.

3. The relation of critical snarks

In [da Silva and Lucchesi 2008][Theorem 3.1], it is shown that for a k-flow-critical graph, the subgraph $G \setminus e$ must also admit a k-flow for every edge e of G. In [Jaeger 1988][Theorem 8.2] it was shown that every 3-edge-connected cubic graph G such that $G \setminus e$ admits a 4-flow, has a 5-flow. In [da Silva and Lucchesi 2007][Theorem 3.1] this result was generalized to show that every k-flow-critical graph admits a $(k+1)$-flow. Da Silva et al. identified the 4-flow-critical snarks among the snarks of order at most 28 generated by Brinkmann et al. (see [da Silva et al. 2013] and [Brinkmann et al. 2013]). They found out that less than 5% of those are in fact 4-flow-critical. On the other hand, they observed that every non-4-flow-critical snark G has a 4-flow-critical snark H as a minor. Minor H does have a 5-flow. Then, an approach towards solving the 5-Flow Conjecture could be to try to extend the 5-flow of minor H to G. That reasoning was the inspiration to this work. An intuition that the task could be easier when minor H is close to being Hamiltonian made us focus on studying hypohamiltonian snarks first.

Given a cubic graph G, the graph $G \setminus e$ is a subdivision of another cubic graph G_e, which is called the *underlying cubic graph* of $G \setminus e$. It is well known that any graph G admits a k-flow if and only if any of its subdivisions has a k-flow (see [de Almeida e Silva 1991][Redução 3.2]). The following proposition follows from these observations and Theorem 1.2.

Proposition 3.1 *Let G be a snark and e be an edge of G. Snark G is 4-flow-critical if and only if the underlying cubic graph G_e of $G \setminus e$ is 3-edge-colourable for any edge e of G.*

Many famous snarks such as the Petersen graph, the first and second Blanuša, the first and second Loupekine, the Szekeres snark and the Double-star snark, are both hypohamiltonian and 4-flow-critical. Moreover, the infinite family of flower-snarks is also known to be both hypohamiltonian [Fiorini 1983] and 4-flow-critical [da Silva and Lucchesi 2012]. Steffen showed in [Steffen 1998][Theorem 4.31] that every hypohamiltonian snark is also bicritical. These observations made us conjecture that hypohamiltonian and 4-flow-critical snarks could be the same class, or at least related somehow. It was indeed possible to use the same technique used by Steffen [Steffen 1998][Theorem 4.31] to prove the following proposition.

Proposition 3.2 *If a snark G is hypohamiltonian, then G is 4-flow-critical.*

Proof Let G be a hypohamiltonian snark and $e := (u, v)$ be an edge of G such that $N(u) := \{v, u_1, u_2\}$ and $N(v) := \{u, v_1, v_2\}$. Since $G \setminus v$ is Hamiltonian, the graph $H := G\setminus\{u, v\}$ has an odd Hamiltonian path P whose ends are u_1 and u_2. The edges of H can be 3-edge-coloured by alternating two colours in the Hamiltonian path and assigning the third colour to the remaining edges. Clearly, $H \cup \{(u_1, u_2), (v_1, v_2)\} = G_e$. We can extend the 3-edge-colouring to G_e by assigning the second colour of P not incident with u_1 and u_2 to (u_1, u_2) and the third colour to (v_1, v_2). It follows from Proposition 3.1 that G is 4-flow-critical. ∎

It follows from Proposition 3.2 that every hypohamiltonian snark admits a 5-flow. This observation gives a positive answer to the question proposed in [Cavicchioli et al. 2003, Question 6.1]. The converse of this assertion is not true; there are 4-flow-critical snarks which are not hypohamiltonian. By using a computer program we found 16 counter-examples on 26 vertices. One of them is depicted in Figure 1.

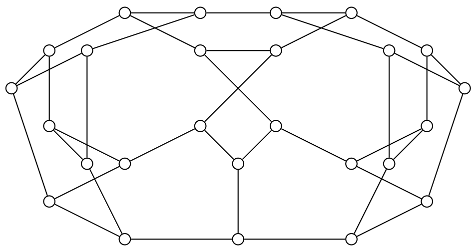


Figure 1. A 4-flow-critical non-hypohamiltonian snark

Cavicchioli et. al. presented the number of snarks of order at most 28 that are hypohamitonian, 2-vertex-critical, 2-vertex-co-critical or bicrital (see [Cavicchioli et al. 2003] [Table 1]). In [da Silva et al. 2013] the number of snarks of order at most 28 that are 4-flow-critical is presented and it is, surprisingly, identical to the number of 2-vertex-critical snarks. That observation led us to conjecture that these two classes should be the same. This conjecture was later proved, as shown next.

Theorem 3.3 *A snark is 4-flow-critical if and only if it is 2-vertex-critical.*

Proof Let G be a snark, and let $e := (u, v)$ be an arbitrary edge of G. Assume that the neighbourhoods of u and v are $N(u) := \{v, u_1, u_2\}$ and $N(v) := \{u, v_1, v_2\}$ (see Figure 2), respectively. Then, the underlying cubic graph G_e of $G \setminus e$ can be obtained from graph $G \setminus \{u, v\}$ by the addition of edges (u_1, u_2) and (v_1, v_2) (see Figure 2).

If snark G is 4-flow-critical, then, by Proposition 3.1, the underlying cubic graph G_e of $G\setminus e$ has a 3-edge-colouring. Then, so does graph $G\setminus\{u, v\}$, a subgraph of G_e. This argument is valid for every pair of adjacent vertices u and v, whence G is 2-vertex-critical.

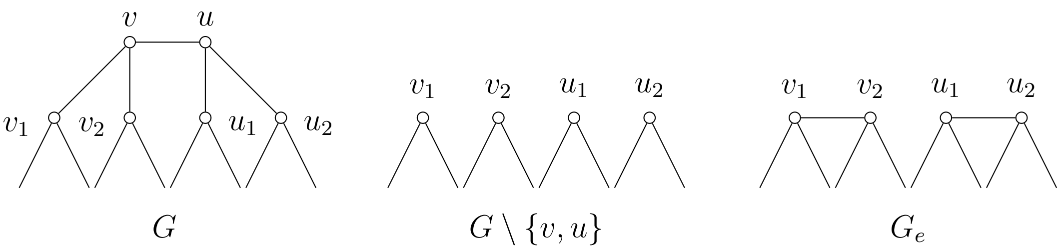
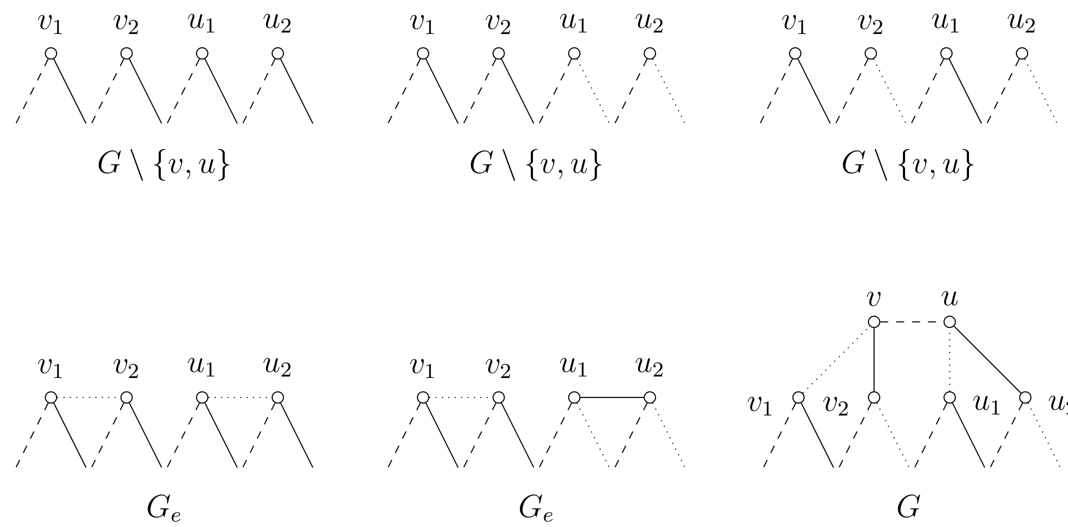


Figure 2. Graph $G \setminus \{u, v\}$ is a subgraph of G_e

Assume now that snark G is 2-vertex-critical. All vertices of $G \setminus \{u, v\}$ have degree three, except $\{u_1, u_2, v_1, v_2\}$, which have degree two. By definition, graph $G \setminus \{u, v\}$ has a 3-edge-colouring. Let M_i, for $i = 1, 2, 3$, be the set of edges with colour i in a 3-edge-colouring of $G \setminus \{u, v\}$. Each set M_i is a (not necessarily perfect) matching and covers an even number of vertices, leaving an also even number of vertices uncovered. However, the only vertices of $G \setminus \{u, v\}$ that may not be covered by some matching M_i are precisely the four vertices of degree two. Hence, at most two sets M_i are not incident with vertices $\{u_1, u_2, v_1, v_2\}$.



(a) First case (b) Second case (c) Third case

Figure 3. Cases considered in the proof of Theorem 3.3

Consider first the case in which precisely one set, say M_1, is not incident with any of the four vertices of degree two. Then, we can add edges (u_1, u_2) and (v_1, v_2) and assign them colour 1 in order to obtain a 3-edge-colouring of G_e (see Figure 3.a). We may thus assume that two sets M_i are not incident with vertices $\{u_1, u_2, v_1, v_2\}$. If one set, say M_1, is not incident with $\{u_1, u_2\}$ and the other, say M_2, is not incident with $\{v_1, v_2\}$, we can add edges (u_1, u_2) and (v_1, v_2) and assign them colours 1 and 2 in order to obtain a 3-edge-colouring of G_e (see Figure 3.b). We may thus assume that each set is not incident with one of $\{u_1, u_2\}$ and one of $\{v_1, v_2\}$. Adjust notation, if necessary, so that M_1 is not incident with $\{u_1, v_1\}$ and M_2 is not incident with $\{u_2, v_2\}$. In this case, we can extend the 3-edge-colouring of $G \setminus \{u, v\}$ to one of snark G by assigning colour 1 to edges (u, u_1) and (v, v_1), colour 2 to edges (u, u_2) and (v, v_2) and colour 3 to edge (u, v) (see Figure 3.c); an obvious contradiction. We therefore conclude that it is always possible to obtain a 3-edge-colouring of G_e, for every edge e of G. By Proposition 3.1, G is 4-flow-critical. ∎

A natural question that arises from Theorem 3.3 is whether or not the classes of bicritical and 4-flow-critical snarks are also the same. In 1996, Nedela and Škoviera [Nedela and Škoviera 1996] asked whether or not every 2-vertex-critical snark is also bicritical. Steffen [Steffen 1999] and, independently, Chladný and Škoviera (not published) found counter-examples for such statement. It follows that there are 4-flow-critical snarks which are not 2-vertex-co-critical. Moreover, according to Cavicchioli et. al. [Cavicchioli et al. 2003][Table II], there are more 2-vertex-co-critical snarks than 2-vertex-critical snarks, whence there is no equivalence between 2-vertex-critical and 2-vertex-co-critical snarks either. Another natural question is whether or not there is an equivalence between snarks of oddness 2 and 4-flow-critical snarks. It is known that the oddness of a cubic graph G equals two if and only if its resistance also equals two [Steffen 1998][Lemma 2.5]. It follows from Proposition 2.1 that every snark has oddness at least two. Next we show that every 4-flow-critical snark has oddness equal to two.

Proposition 3.4 *If G is a 4-flow-critical snark, then $\omega(G) = 2$.*

Proof Let e be an edge of G. Graph G_e has a 3-edge-colouring, and therefore, oddness zero. Let M_e be a perfect matching of G_e . Then $M := M_e \cup \{e\}$ is a perfect matching of G. Graph $G \setminus M$ is a 2-factor of G which can be obtained from $G_e \setminus M_e$ by the subdivision of two edges, say e_1 and e_2. Edges e_1 and e_2 cannot lie in the same cycle of the 2-factor $G_e \setminus M_e$, otherwise G would have oddness zero. We thus conclude that $\omega(G) = 2$. ∎

The converse of Proposition 3.4 is not true. According to [da Silva et al. 2013] the smallest non-4-flow-critical snarks have order 20. We observed that one of them, the snark presented in [da Silva et al. 2013] [Figure 1] and also depicted in Figure 4 has oddness 2. We can 2-colour a Hamiltonian path with ends u_1 and v_1 and assign a third colour to the remaining edges. This assignment is not a proper 3-edge-colouring because of the edges (v_1, v_2) and (u_1, u_2). Since the removal of $\{(v_1, v_2), (u_1, u_2)\}$ yields a 3-edge-colouring of the graph, it follows that it has resistance two, whence oddness two.

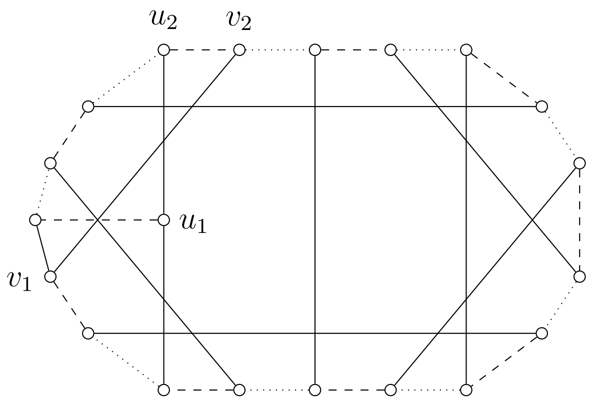


Figure 4. A non-4-flow-critical snark with resistance and oddness two

4. Concluding remarks

We showed that a snark is 4-flow-critical if and only if it is 2-vertex-critical. We also showed that every hypohamiltonian snark is 4-flow-critical and that every 4-flow-critical snark has oddness two. Moreover, we observed that, aside from those, no other relation between the classes of critical snarks studied exist. As described in the text, we came to this conclusion by a combination of research of the literature to find out relations that were known to be false as well as an investigation of counter-examples for those which we could not find information in the literature.

One of the relations observed, specifically that every hypohamiltonian snark is 4-flow-critical, implies that every hypohamiltonian snark has a 5-flow. Such observation allowed us to answer a question proposed by Cavicchioli et al. in 2003.

5. Authors contribution

Breno Lima de Freitas is an undergraduate student at UFSCar and was advised in this work by Cândida Nunes da Silva. When Breno expressed to his advisor that he would like to do research on graph theory, she presented to him her recent discoveries about 4-flow-critical snarks, i. e., that every snark had a 4-flow-critical snark as a minor, and that minor was known to have a 5-flow. Then his advisor gave him as a tentative project, the vague idea of trying to find 4-flow-critical snarks that had a high circumference, as it seemed easier to extend the 5-flow of a 4-flow-critical minor H of a non-4-flow-critical snark G if H had high circumference.

It was Breno who transformed this vague idea into a well defined project by choosing the family of hypohamiltonian snarks to start with and researching interesting properties of those graphs that could be helpful in establishing some relation with 4-flow-critical snarks. In his search he found the survey on snarks by Cavicchioli et al. [Cavicchioli et al. 2003] which brought to his attention two important pieces of information: (i) the number of hypohamiltonian snarks of order at most 28; (ii) and the open question whether every hypohamiltonian snark has a 5-flow. Breno then brought this information to his advisor, together with the conjecture that hypohamiltonian snarks might be 4-flow-critical and, if so, the open question was solved. Soon after his observation, he and his advisor verified he was right and his project became to follow on the investigation on how the classes of critical snarks are related. Therefore, Breno's contribution to the development of this project was a fundamental one. He showed to have important research skills such as finding closely related references and asking an important research question based on tuned intuition.

The third author, Cláudio L. Lucchesi, contributed to this project by helping to prove Breno and Cândida's conjecture that the classes of 2-vertex-critical and of 4-flow-critical snarks were the same (Theorem 3.3).

The development of this project started in 2013. It was paused during 2014, when Breno received a scholarship from the "Science without borders" program to study for one year at the University of Toronto, in Canada. In 2015 the development of the project was resumed.

References

[Bondy and Murty 1976] Bondy, J. A. and Murty, U. S. R. (1976). *Graph Theory with Applications*. Elsevier North Holland.

[Bondy and Murty 2008] Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*. Springer.

[Brinkmann et al. 2013] Brinkmann, G., Goedgebeur, J., Hägglund, J., and Markstrom, K. (2013). Generation and properties of snarks. *J. Comb. Theory, Series B*, 103(4):468–488. Preliminary version available at `http://arxiv.org/abs/1206.6690`.

[Cavicchioli et al. 2003] Cavicchioli, A., Murgolo, T. E., Ruini, B., and Spaggiari, F. (2003). Special classes of snarks. *Acta Applicandae Mathematica*, 76:57–88.

[Chetwynd and Wilson 1981] Chetwynd, A. G. and Wilson, R. J. (1981). Snarks and Supersnarks. In Alavi, Y., Chartrand, G., Goldsmith, D. L., Lesniak-Foster, L., and Lick, D. R., editors, *The Theory and Applications of Graphs*, pages 215–241. Wiley.

[da Silva and Lucchesi 2007] da Silva, C. N. and Lucchesi, C. L. (2007). Flow-critical graphs. Technical report, Instituto de Computação, UNICAMP, Brasil.

[da Silva and Lucchesi 2008] da Silva, C. N. and Lucchesi, C. L. (2008). Flow-critical graphs. *Electronic Notes in Discrete Mathematics*, 30:165–170.

[da Silva and Lucchesi 2012] da Silva, C. N. and Lucchesi, C. L. (2012). Flower-snarks are flow-critical. Available at `http://www.dcomp.sor.ufscar.br/candida/publications`.

[da Silva et al. 2013] da Silva, C. N., Pesci, L., and Lucchesi, C. L. (2013). Snarks and flow-critical graphs. *Electronic Notes in Discrete Mathematics*, 44:299–305.

[de Almeida e Silva 1991] de Almeida e Silva, L. M. (1991). Fluxos inteiros em grafos. Master's thesis, Departamento de Ciência da Computação - UNICAMP. Em português.

[Diestel 1996] Diestel, R. (1996). *Graph Theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag.

[Fiorini 1983] Fiorini, S. (1983). Hypohamiltonian snarks. Graphs and other combinatorial topics, Proc. 3rd Czech. Symp., Prague 1982, Teubner-Texte Math. 59, 70-75 (1983).

[Fulkerson 1971] Fulkerson, D. (1971). Blocking and anti-blocking pairs of polyhedra. *Math. Programming 1*, 69:168–194.

[Gardner 1976] Gardner, M. (1976). Matematical games: Snarks, boojums and other conjectures related to the four-color-map theorem. *Scientific American*, 234:126–130.

[Isaacs 1975] Isaacs, R. (1975). Infinite families of non-trivial trivalent graphs which are not tait colorable. *Amer. Math. Monthly*, 82:221–239.

[Jaeger 1988] Jaeger, F. (1988). Nowhere-zero flow problems. volume 3 of *Selected Topics in Graph Theory*, pages 71–95. Academic Press.

[Nedela and Škoviera 1996] Nedela, R. and Škoviera, M. (1996). Decompositions and reductions of snarks. *Journal of Graph Theory*, 22:253–279.

[Seymour 1979] Seymour, P. D. (1979). Sums of circuits. *Graph theory and related topics*, 1:341–355.

[Seymour 1981] Seymour, P. D. (1981). Nowhere-zero 6-flows. *J. Comb. Theory, Series B*, 30:130–135.

[Seymour 1995] Seymour, P. D. (1995). Nowhere-zero flows. In Graham, R. L. and M. Grötschel, L. L., editors, *Handbook of Combinatorics*, chapter 4, pages 289–299. Elsevier.

[Steffen 1998] Steffen, E. (1998). Classifications and characterizations of snarks. *Discrete Mathematics*, 188:183–203.

[Steffen 1999] Steffen, E. (1999). Non-bicritical critical snarks. *Graphs Comb.*, 15:473–480.

[Szekeres 1973] Szekeres, G. (1973). Polyhedral decompositions of cubic graphs. *Bull. Austral. Math. Soc.*, 8:367 – 387.

[Tutte 1954] Tutte, W. T. (1954). A contribution to the theory of chromatic polynomials. *Can. J. Math.*, 6:80–91.

[West 1996] West, D. B. (1996). *Introduction to Graph Theory*. Prentice Hall.

[Younger 1983] Younger, D. H. (1983). Integer flows. *Journal of Graph Theory*, 7:349–357.

[Zhang 1997] Zhang, C.-Q. (1997). *Integer Flows and Cycle Covers of Graphs*. Marcel Dekker.