Decomposition of $(2k+1)$-regular graphs containing special spanning $2k$-regular Cayley Graphs into paths of length $2k+1$ *

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Abstract. *A P_ℓ-decomposition of a graph G is a set of paths with ℓ edges in G that cover the edge set of G. Favaron, Genest, and Kouider (2010) conjectured that every $(2k+1)$-regular graph that contains a perfect matching admits a P_{2k+1}-decomposition. They also verified this conjecture for 5-regular graphs without cycles of length 4. In 2015, Botler, Mota, and Wakabayashi extended this result to 5-regular graphs without triangles. In this paper, we verify this conjecture for $(2k+1)$-regular graphs that contain the k-th power of a spanning cycle; and for 5-regular graphs that contain certain spanning 4-regular Cayley graphs.*

Resumo. *Uma P_ℓ-decomposição de um grafo G é um conjunto de caminhos aresta-disjuntos com ℓ arestas em G que cobre o conjunto de arestas de G. Favaron, Genest, e Kouider (2010) conjecturaram que todo grafo $(2k+1)$-regular que contém um emparelhamento perfeito admite uma P_{2k+1}-decomposição. Eles também verificaram essa conjectura para grafos 5-regulares sem ciclos de comprimento 4. Em 2015, Botler, Mota, e Wakabayashi estenderam esse resultado para grafos 5-regulares sem triângulos. Neste artigo, verificamos essa conjectura para grafos $(2k+1)$-regulares que contêm uma k-ésima potência de um ciclo gerador; e para grafos 5-regulares que contêm alguns tipos de grafos de Cayley 4-regulares e geradores.*

1. Introduction

All graphs in this paper are simple. We denote by P_ℓ the path with ℓ edges, and a *P_ℓ-decomposition* of a graph G is a set $\mathcal{D}$ of edge-disjoint copies of P_ℓ in G that cover its edge set. Kotzig (1957) and Bouchet and Fouquet (1983) proved that a 3-regular graph G admits a P_3-decomposition if and only if G contains a perfect matching. Favaron, Genest, and Kouider (2010) extended this result by proving that every 5-regular graph that contains a perfect matching and no cycles of length 4 admits a P_5-decomposition; and proposed the following conjecture.

Conjecture 1 (Favaron–Genest–Kouider, 2010)**.** *If G is a $(2k+1)$-regular graph that contains a perfect matching, then G admits a P_{2k+1}-decomposition.*

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In 2015, Botler, Mota, and Wakabayashi verified Conjecture 1 for triangle-free 5-regular graphs, and, in 2017, Botler, Mota, Oshiro, and Wakabayashi generalized this result for $(2k+1)$-regular graphs with girth at least $2k$.

A 5-regular graph contains a perfect matching if and only if it contains a spanning 4-regular graph. In this paper, we explore Conjecture 1 for $(2k+1)$-regular graphs that contain special spanning $2k$-regular graphs as follows. Throughout the text, Γ denotes a finite group of order n; $+$ denotes the group operation of Γ; and 0 denotes the identity of Γ. For each $x \in \Gamma$, we denote by $-x$ the element $y \in \Gamma$ such that $x + y = 0$, and the operation $-$ denotes the default binary operation $-\colon (x, y) \mapsto x + (-y)$. Let $S \subseteq \Gamma$ be a set not containing the identity of Γ, and such that $-x \in S$ for every $x \in S$. The *Cayley graph* $X(\Gamma, S)$ is the graph H with $V(H) = \Gamma$, and $E(H) = \{xy\colon y - x \in S\}$ (see [Godsil and Royle 2013]). We allow S not to be a set generating Γ, which implies $X(\Gamma, S)$ not to be necessarily connected. We say that H is *simply commutative* if (i) $x + y = y + x$ for every $x, y \in S$, and (ii) $x + y = 0$ for $x, y \in S$ only if $y = -x \neq x$. Since $0 \notin S$, condition (ii) guarantees that H is a simple graph. In such a graph, the neighborhood of a vertex $v \in \Gamma$ is $N(v) = \{x + v\colon x \in S\}$. Although the definition of Cayley graphs can be made more general, we consider only non-directed simple graphs in order to tackle Conjecture 1.

We present two results regarding Conjecture 1 for graphs that contain spanning Cayley graphs. We verify it for $(2k+1)$-regular graphs that contain the k-th power of a spanning cycle (see Section 2); and for 5-regular graphs that contain spanning simply commutative 4-regular Cayley graphs (see Section 3). Due to space limitations, we present only sketches of some proofs.

2. Regular graphs that contain spanning powers of cycles

Given positive integers k and n, the k-th power of the cycle on n vertices, denoted by C_n^k, is the graph on the vertex set $\{0, \ldots, n-1\}$ such that, for every vertex v, we have $x \in N(v)$ if and only if $x = v + r \pmod{n}$, where $r \in \{-k, \ldots, -1\} \cup \{1, \ldots, k\}$. Given a perfect matching M of a graph G we say that a P_ℓ-decomposition $\mathcal{D}$ of a graph G is *M-centered* if for every $P = a_0a_1 \ldots a_ia_{i+1} \ldots a_{\ell-1}a_\ell \in \mathcal{D}$, we have $a_ia_{i+1} \in M$ for $i = (\ell-1)/2$. The next results are examples of M-centered decomposition.

Proposition 1. *If G is a 5-regular graph that contains a spanning copy K of $K_{4,4}$, and $M = G - E(K)$, then G admits an M-centered P_5-decomposition.*

Proof. Let (R, L) be the bipartition of K, where $R = \{r_1, r_2, r_3, r_4\}$ and $L = \{l_1, l_2, l_3, l_4\}$. Since K is a complete bipartite graph, if $xy \in M$, then either $x, y \in R$ or $x, y \in L$. Thus, we may suppose, without loss of generality, that $M = \{r_1r_2, r_3r_4, l_1l_2, l_3l_4\}$. Therefore, $\{l_1r_1l_3l_4r_2l_2,\ l_3r_3l_1l_2r_4l_4,\ r_1l_2r_3r_4l_1r_2,\ r_3l_4r_1r_2l_3r_4\}$ is an M-centered decomposition of G, as desired. □

The following proposition is the main result of this section.

Proposition 2. *Let $n, k \in \mathbb{N}$ with $k < n/2$. If G is a $(2k+1)$-regular graph on n vertices that contains a copy C of C_n^k, and $M = G - E(C)$, then G admits an M-centered P_{2k+1}-decomposition.*

Proof. Let $V(C) = \{0, \ldots, n-1\}$ be as above. Since C is a $2k$-regular graph, M is a perfect matching of G. Given $i \in V(C)$, let Q_i be the path $v_0v_1 \ldots v_k$ in which $v_0 = i$;

and, for $j = 1, \dots, k$, we have $v_j = v_{j-1} + j$ if j is odd; and $v_j = v_{j-1} - j$ if j is even. Note that for every $j = 1, \dots, k$, the path Q_i contains an edge xy such that $|x - y| = j$, and hence the set $\mathcal{Q} = \big\{Q_i \colon i \in V(C)\big\}$ is a P_k-decomposition of C. Also, we have $V(Q_i) = \big\{i+r \pmod n \colon r \in \{-\lfloor k/2 \rfloor, -\lfloor k/2 \rfloor + 1, \dots, \lceil k/2 \rceil\}\big\}$. Now, given an edge $e = ij \in M$, let $P_e = Q_i \cup \{ij\} \cup Q_j$. Since Q_i and Q_j have, respectively, i and j as end vertices, and $E(Q_i) \cap E(Q_j) = \emptyset$, the graph P_e is a trail of length $2k + 1$. Therefore, the set $\mathcal{D} = \{P_e \colon e \in M\}$ is a decomposition of G into trails of length $2k + 1$. We claim that, in fact, $\mathcal{D}$ is a P_{2k+1}-decomposition of G. For that, we prove that if $ij \in M$, then $V(Q_i) \cap V(Q_j) = \emptyset$. Indeed, note that for every $e = ij \in M$, we have $|i - j| > k$, otherwise $ij \in E(C)$. Now, suppose that there is a vertex v in $V(Q_i) \cap V(Q_j)$. Then, there are r_1, r_2 with $-\lfloor k/2 \rfloor \leq r_1, r_2 \leq \lceil k/2 \rceil$, and such that $i + r_1 = v = j + r_2$. Suppose, without loss of generality, that $i > j$. Then, we have $r_2 - r_1 = i - j > k$, but $r_2 - r_1 \leq \lceil k/2 \rceil + \lfloor k/2 \rfloor = k$, a contradiction. □

Naturally, Propositions 1 and 2 can be extended to the following result, which is used in the proof of Theorem 1.

Corollary 1. *If G is a 5-regular graph that contains a 4-factor K in which every component is isomorphic to $K_{4,4}$ or to the 2nd power of a cycle, and $M = G - E(K)$, then G admits an M-centered P_5-decomposition.*

3. 5-regular graphs that contain special spanning Cayley graphs

In this section, we explore 5-regular graphs that contain spanning simply commutative 4-regular Cayley graphs. Given two elements g, r of Γ, we say that $\{g, r\}$ is a *simple commutative generator* (SCG) if (i) $0 \notin \{g, r, 2g, 2r\}$; (ii) $g \notin \{r, -r\}$; and (iii) $g + r = r + g$. Let $S = \{g, -g, r, -r\}$, and consider the Cayley graph $C = X(\Gamma, S)$. By construction, C is a simply commutative Cayley graph (see Section 1). Conditions (i) and (ii) guarantee that C is a simple graph, while condition (iii) introduces the main restriction explored in this paper. In this case, we say that C is the graph *generated* by $\{g, r\}$, and that $\{g, r\}$ is the *generator* of C. Given an SCG $\{g, r\}$, we say that a simple 5-regular graph G is a *$\{g, r\}$-graph* if G contains a spanning Cayley graph generated by $\{g, r\}$. In this case, we denote the Cayley graph by $C_{g,r}$, and the perfect matching $G - E(C_{g,r})$ by $M_{g,r}$. Given a vertex $u \in V(G)$, we say that an edge $uv \in E(G)$ is a *green* (resp. *red*) *out-edge* of u if $v = u + g$ (resp. $v = u + r$).

In 2015, Botler, Mota, and Wakabayashi verified Conjecture 1 for triangle-free 5-regular graphs. For that they (i) find an initial decomposition of G into paths and trails; and (ii) perform some exchanges of edges between the elements of $\mathcal{D}$, preserving a special invariant, while minimizing the number of trails that are not paths. The general case of the proof of our main theorem follows this framework, but consists of three steps. First, from the structure of Cayley graphs, we find an initial decomposition into trails of length 5. Then, we show how to exchange edges to obtain a decomposition in which the *bad elements* (the trails that are not paths) are distributed in circular fashions. Finally, we show how to deal with these "cycles of bad elements".

Theorem 1. *Every $\{g, r\}$-graph admits a P_5-decomposition.*

Skecth of the proof. First, if $g = 2r$ (or $r = 2g$), then each component of $C_{g,r}$ is a 2nd power of a cycle. Also, one can prove that if $2g + 2r = 0$ and $2g - 2r = 0$, then each component of $C_{g,r}$ is a copy of $K_{4,4}$. In these cases, the statement follows by Corollary 1.

Thus, we assume that $g \neq 2r$, $r \neq 2g$, $2g + 2r \neq 0$ and $2g - 2r \neq 0$. Now, for each $e = uv \in M_{g,r}$, let $T_e = x'xuvyy'$ be the trail in which ux and vy are the green out-edges of u and v, respectively, and xx' and yy' are the red out-edges of x and y, respectively. It is not hard to check that $\mathcal{D} = \{T_e : e \in M_{g,r}\}$ is a decomposition of G. One can prove that, since G is a $\{g, r\}$-graph, an element $T = x'xuvyy'$ of $\mathcal{D}$ is either a path, or $uvyy'$ is a triangle, and T has no other cycle (see Figure 1). In the later case, we say that T is a *bad element*, and that u and v are the *connection vertices* of T, and y is the *tricky vertex* of T. If the tricky vertex of a bad element T is not a connection vertex of any other element, then we say that T is a *free* bad element. By repeatedly exchanging edges between a free bad element and a different element of the decomposition, we obtain a new decomposition $\mathcal{D}'$ of G into paths of length 5 and bad elements that are not free. Then, these bad elements are distributed in circular fashions, which we call *bad chains*. Now, we perform an extra exchange of edges in order to convert a bad element into a path, "breaking" a bad chain, and allowing us to perform the exchanges of the second step. $\square$

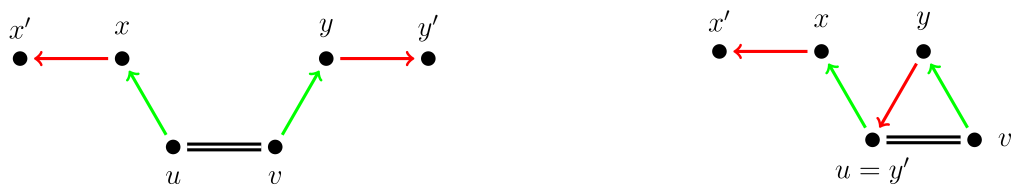

Figure 1. A path and a trail of length 5.

4. Concluding remarks

Due to the underlying group structure, we believe that the techniques developed here can be extended to $(2k+1)$-regular graphs that contain more general spanning Cayley graphs, or that contain special spanning *Schreier Graphs*, which could give us significant insights to solving the general case of Conjecture 1 (see [Gross 1977]).

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