A new sufficient condition for the existence of 3-kernels

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Abstract. Let $D$ be a digraph and $k$ be a positive integer. We say a subset $N$ of $V(D)$ is a $k$-kernel of $D$ if it is both $k$-independent and $(k-1)$-absorbent. A short chord of a closed trail $C = (v_0, v_1, \ldots, v_t)$ is an arc $a = (v_i, v_j)$ which does not belong to $C$ and the distance from $v_i$ to $v_j$ in $C$ is exactly two. The spacing between two chords $e = (u, v)$ and $f = (x, y)$ in $C$ is the distance from $u$ to $x$ in $C$. A set of chords in a closed trail $C$ has an odd spacing if at least two chords have an odd spacing.

In this work, we prove that if $D$ is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then $D$ has a 3-kernel.

1. Introduction

We assume that all digraphs have no loops or multiple arcs. The vertex set of a digraph $D$ is denoted by $V(D)$ and its arc set by $A(D)$. All trails, paths and cycles are considered to be directed. An arc $(u, v) \in A(D)$ is symmetric if $(v, u) \in A(D)$. A digraph $D$ is strongly connected if for every pair of vertices $u, v \in V(D)$, there exists a path from $u$ to $v$ in $D$. The distance between two vertices $u$ and $v$ in a digraph $D$, denoted by $d_D(u, v)$, is the length of the shortest path from $u$ to $v$ in $D$. We say that a closed trail or a cycle is even (resp. odd) if its length is even (resp. odd). For undefined notation, we refer the reader to [Bondy and Murty 2008].

A chord of a closed trail $C = (c_0, \ldots, c_{n-1}, c_0)$ is an arc $a = (c_i, c_j)$, where $c_i, c_j \in V(C)$ but $a \notin A(C)$. If the distance from $c_i$ to $c_j$ in $C$ is two, we say that it is a short chord. The spacing between two chords $e = (u, v)$ and $f = (x, y)$ of $C$ is the distance from $u$ to $x$ in $C$. A set of chords in a closed trail $C$ has an odd spacing if at least two chords have an odd spacing. Figure 1 illustrates a cycle with two odd spaced short chords.

Figure 1. A cycle with two odd spaced short chords.
A kernel of a digraph $D$ is a set $K \subseteq V(D)$ which is independent in $D$ and for every vertex $u \in V(D) \setminus K$, there exists an arc $(u, v)$, where $v \in K$. A digraph is kernel-perfect if every induced subdigraph has a kernel. A subset $N$ of $V(D)$ is $k$-independent if for every pair of vertices $u, v \in N$, the distance from $u$ to $v$ in $D$ is at least $k$; also, $N$ is $\ell$-absorbent if for every vertex $u \in V(D) \setminus N$, there exists $v \in N$ such that the distance from $u$ to $v$ is at most $\ell$. We say that a subset $N$ of $V(D)$ is a $(k, \ell)$-kernel of $D$ if it is both $k$-independent and $\ell$-absorbent. A $k$-kernel is a $(k, k-1)$-kernel and a kernel is a 2-kernel.

The concept of kernel was introduced by von Neumann and Morgenstern in 1944 [von Neumann and Morgenstern 1944] in the context of game theory to model social and economic interactions. Richardson [Richardson 1946] proved that every digraph which has no odd cycle has a kernel, a seminal result of the field. In light of Richardson’s Theorem, kernel theory gained a lot of attention and was thoroughly researched, yielding new results regarding the existence of kernels in digraphs with odd cycles. In 1980, Kwasnik [Kwasnik 1980] proposed the concept of $k$-kernels and later generalized Richardson’s Theorem to $k$-kernels [Kwasnik 1981].

In this work, we prove the following.

**Theorem 1.** Let $D$ be a digraph. If $D$ is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then $D$ has a 3-kernel.

2. A sufficient condition for the existence of 3-kernels in digraphs

In this section, we prove Theorem 1. An useful tool for demonstrating the existence of 3-kernels in digraphs is Lemma 3 which states that a digraph has a 3-kernel if, and only if, its 2-closure has a kernel. Before we present Lemma 3 we introduce the definition of the $k$-closure of a digraph.

**Definition 2.** Let $D$ be a digraph. The $k$-closure of $D$, denoted by $C^k(D)$, is the digraph $D'$, where $V(D') = V(D)$ and $(u, v) \in A(D')$ if $d_D(u, v) \leq k$. Figure 2 illustrates an example of a digraph and its 2-closure.

**Lemma 3 ([Galeana-Sánchez and Hernández-Cruz 2014]).** Let $k \geq 3$ be an integer. Let $D$ be a digraph and let $K \subseteq V(D)$. The subset of vertices $K$ is a $k$-kernel of $D$ if, and only if, $K$ is a kernel of $C^{(k-1)}(D)$.

A particularly useful theorem was proved by Duchet in 1980.
Theorem 4 ([Duchet 1980]). If every cycle of a digraph $D$ has a symmetric arc, then $D$ is kernel-perfect.

The strategy of the proof to our theorem is to show that the 2-closure of a digraph $D$ that satisfies the hypothesis from Theorem 1 has a symmetric arc in every cycle. From Theorem 4, the 2-closure of $D$ has a kernel; therefore, by Lemma 3, the digraph has a 3-kernel.

Before we present the demonstration, we must introduce the main lemma used in the proof.

Lemma 5. Let $D$ be a strongly connected digraph. If every odd cycle in $D$ has a short chord and every even closed trail in $D$ has three short chords with an odd spacing, then for every $(u, v) \in A(D)$ there is a path of length at most two from $v$ to $u$ in $D$.

Proof. Let $f = (u, v) \in A(D)$. Since $D$ is strongly connected, there is a minimal path $T = (t_0 = v, \ldots, t_s = u)$ from $v$ to $u$ in $D$. Note that $C = T \cup (u, v)$ is a cycle. Clearly, if $|C| = 3$, then $|T| = 2$ and the result follows. Assume, for the sake of contradiction, that $|T| > 2$ and $|C|$ is even. Since $T$ is minimal, the only possible short chords in $C$ are $(t_{s-1}, v)$ and $(u, t_1)$, which contradicts the hypothesis of the existence of three short chords in $C$. Assume then, for the sake of contradiction, that $|T| > 2$ and $|C|$ is odd. Due to the hypothesis and the minimality of $T$, $C$ has a short chord: $(t_{s-1}, v)$ or $(u, t_1)$. Let $(a, b)$ be one of such chords. Note that $(a, b) \cup (b, t, a)$ is a cycle of even length and, therefore, must have three short chords. Analogously to the former case, one of the chords connects two vertices in $T$, a contradiction. Hence, $|T| \leq 2$. $lacksquare$

We are now ready to demonstrate Theorem 1.

Proof of Theorem 1. Let $D' = C(D)$. It follows from Lemma 3 that $D$ has a 3-kernel if $D'$ has a kernel. We show that every cycle in $D'$ has a symmetric arc and, by Theorem 4, $D'$ has a kernel.

Assume that there exists a cycle $C = (c_0, c_1, \ldots, c_{n-1}, c_0)$ in $D'$ with no symmetric arc. Choose such cycle $C$ with the shortest length in $D'$. Note that no arc in $C$ exists in $D$, because it would be symmetric: from Lemma 5, if $(c_i, c_{i+1}) \in A(D)$, then $(c_{i+1}, c_i) \in A(D')$. Therefore, $d_D(c_i, c_{i+1}) = 2$, for every $i \leq n$ (notation modulo $n$).

Since every arc $(c_i, c_{i+1})$ of $C$ is in $A(D')$ – but not in $A(D)$ – for every arc $(c_i, c_{i+1})$ in $C$ there exists a vertex $c_{i,(i+1)} \in V(D)$ such that $(c_i, c_{(i+1)}, c_{i+1})$ exists in $D$. Let $C'$ be the closed trail resulting from the substitution of every $(c_i, c_{i+1})$ in $C$ for $(c_i, c_{i,(i+1)}, c_{i+1})$. Figure 3 illustrates an example of $C$ and $C'$.

Since $C$ is a cycle, every vertex in $C$ is distinct. Also, note that $C'$ is an even closed trail and, by the hypothesis, it has three short chords with an odd spacing. Since every vertex in $C'$ that is in $C$ has the same index parity and the three short chords of $C'$ have an odd spacing, one of the short chords in $C'$ must be one of the arcs of $C$. Given that such chord exists in $A(D)$, it is symmetric in $D'$ from Lemma 5. Therefore, $C$ has a symmetric arc.

Since every cycle of $D'$ has at least one symmetric arc, it follows from Theorem 4 that $D'$ has a kernel. Hence, from Lemma 3, $D$ has a 3-kernel. $lacksquare$
Figure 3. An illustration of a cycle $C$, in dashed red, and a closed trail $C'$, in black.

References


