A new sufficient condition for the existence of 3-kernels

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Abstract. *Let D be a digraph and k be a positive integer. We say a subset N of $V(D)$ is a k-**kernel** of D if it is both k-independent and $(k-1)$-absorbent. A **short chord** of a closed trail $C = (v_0, v_1, \ldots, v_t)$ is an arc $a = (v_i, v_j)$ which does not belong to C and the distance from v_i to v_j in C is exactly two. The **spacing** between two chords $e = (u, v)$ and $f = (x, y)$ in C is the distance from u to x in C. A set of chords in a closed trail C has an **odd spacing** if at least two chords have an odd spacing.*

In this work, we prove that if D is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then D has a 3-kernel.

1. Introduction

We assume that all digraphs have no loops or multiple arcs. The vertex set of a digraph D is denoted by $V(D)$ and its arc set by $A(D)$. All trails, paths and cycles are considered to be directed. An arc $(u, v) \in A(D)$ is **symmetric** if $(v, u) \in A(D)$. A digraph D is **strongly connected** if for every pair of vertices $u, v \in V(D)$, there exists a path from u to v in D. The **distance** between two vertices u and v in a digraph D, denoted by $d_D(u, v)$, is the length of the shortest path from u to v in D. We say that a closed trail or a cycle is **even** (resp. **odd**) if its length is even (resp. odd). For undefined notation, we refer the reader to [Bondy and Murty 2008].

A **chord** of a closed trail $C = (c_0, \ldots, c_{n-1}, c_0)$ is an arc $a = (c_i, c_j)$, where $c_i, c_j \in V(C)$ but $a \notin A(C)$. If the distance from c_i to c_j in C is two, we say that it is a **short chord**. The **spacing** between two chords $e = (u, v)$ and $f = (x, y)$ of C is the distance from u to x in C. A set of chords in a closed trail C has an **odd spacing** if at least two chords have an odd spacing. Figure 1 illustrates a cycle with two odd spaced short chords.

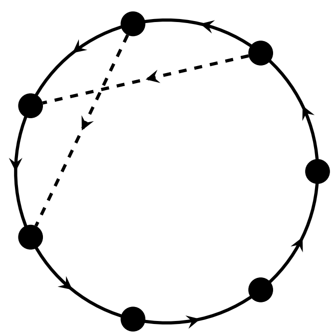

Figure 1. A cycle with two odd spaced short chords.

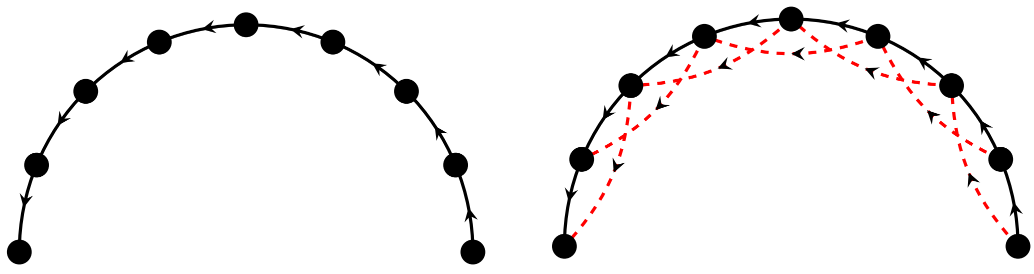

Figure 2. An example of a digraph D and its 2-closure $C^2(D)$. Those arcs in $C^2(D)$ which are not in D are dashed and painted red.

A **kernel** of a digraph D is a set $K \subseteq V(D)$ which is independent in D and for every vertex $u \in V(D) \setminus K$, there exists an arc (u, v), where $v \in K$. A digraph is **kernel-perfect** if every induced subdigraph has a kernel. A subset N of $V(D)$ is k**-independent** if for every pair of vertices $u, v \in N$, the distance from u to v in D is at least k; also, N is ℓ**-absorbent** if for every vertex $u \in V(D) \setminus N$, there exists $v \in N$ such that the distance from u to v is at most ℓ. We say that a subset N of $V(D)$ is a (k, ℓ)**-kernel** of D if it is both k-independent and ℓ-absorbent. A k**-kernel** is a $(k, k-1)$-kernel and a **kernel** is a 2-kernel.

The concept of kernel was introduced by von Neumann and Morgenstern in 1944 [von Neumann and Morgenstern 1944] in the context of game theory to model social and economic interactions. Richardson [Richardson 1946] proved that every digraph which has no odd cycle has a kernel, a seminal result of the field. In light of Richardson's Theorem, kernel theory gained a lot of attention and was thoroughly researched, yielding new results regarding the existence of kernels in digraphs with odd cycles. In 1980, Kwasnik [Kwaśnik 1980] proposed the concept of k-kernels and later generalized Richardson's Theorem to k-kernels [Kwaśnik 1981].

In this work, we prove the following.

Theorem 1. *Let D be a digraph. If D is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then D has a 3-kernel.*

2. A sufficient condition for the existence of 3-kernels in digraphs

In this section, we prove Theorem 1. An useful tool for demonstrating the existence of 3-kernels in digraphs is Lemma 3, which states that a digraph has a 3-kernel if, and only if, its 2-closure has a kernel. Before we present Lemma 3, we introduce the definition of the k-closure of a digraph.

Definition 2. Let D be a digraph. The k**-closure** of D, denoted by $C^k(D)$, is the digraph D', where $V(D') = V(D)$ and $(u, v) \in A(D')$ if $d_D(u, v) \leq k$. Figure 2 illustrates an example of a digraph and its 2-closure.

Lemma 3 ([Galeana-Sánchez and Hernández-Cruz 2014]). *Let $k \geq 3$ be an integer. Let D be a digraph and let $K \subseteq V(D)$. The subset of vertices K is a k-kernel of D if, and only if, K is a kernel of $C^{(k-1)}(D)$.*

A particularly useful theorem was proved by Duchet in 1980.

Theorem 4 ([Duchet 1980]). *If every cycle of a digraph D has a symmetric arc, then D is kernel-perfect.*

The strategy of the proof to our theorem is to show that the 2-closure of a digraph D that satisfies the hypothesis from Theorem 1 has a symmetric arc in every cycle. From Theorem 4, the 2-closure of D has a kernel; therefore, by Lemma 3 the digraph has a 3-kernel.

Before we present the demonstration, we must introduce the main lemma used in the proof.

Lemma 5. *Let D be a strongly connected digraph. If every odd cycle in D has a short chord and every even closed trail in D has three short chords with an odd spacing, then for every $(u, v) \in A(D)$ there is a path of length at most two from v to u in D.*

Proof. Let $f = (u, v) \in A(D)$. Since D is strongly connected, there is a minimal path $T = (t_0 = v, \dots, t_s = u)$ from v to u in D. Note that $C = T \cup (u, v)$ is a cycle. Clearly, if $|C| = 3$, then $|T| = 2$ and the result follows. Assume, for the sake of contradiction, that $|T| > 2$ and $|C|$ is even. Since T is minimal, the only possible short chords in C are (t_{s-1}, v) and (u, t_1), which contradicts the hypothesis of the existence of three short chords in C. Assume then, for the sake of contradiction, that $|T| > 2$ and $|C|$ is odd. Due to the hypothesis and the minimality of T, C has a short chord: (t_{s-1}, v) or (u, t_1). Let (a, b) be one of such chords. Note that $(a, b) \cup (b, T, a)$ is a cycle of even length and, therefore, must have three short chords. Analogously to the former case, one of the chords connects two vertices in T, a contradiction. Hence, $|T| \leq 2$. ■

We are now ready to demonstrate Theorem 1.

Proof of Theorem 1. Let $D' = C^2(D)$. It follows from Lemma 3 that D has a 3-kernel if D' has a kernel. We show that every cycle in D' has a symmetric arc and, by Theorem 4, D' has a kernel.

Assume that there exists a cycle $C = (c_0, c_1, \dots, c_{n-1}, c_0)$ in D' with no symmetric arc. Choose such cycle C with the shortest length in D'. Note that no arc in C exists in D, because it would be symmetric: from Lemma 5, if $(c_i, c_{i+1}) \in A(D)$, then $(c_{i+1}, c_i) \in A(D')$. Therefore, $d_D(c_i, c_{i+1}) = 2$, for every $i \leq n$ (notation modulo n).

Since every arc (c_i, c_{i+1}) of C is in $A(D')$ – but not in $A(D)$ – for every arc (c_i, c_{i+1}) in C there exists a vertex $c_{i,(i+1)} \in V(D)$ such that $(c_i, c_{i,(i+1)}, c_{i+1})$ exists in D. Let C' be the closed trail resulting from the substitution of every (c_i, c_{i+1}) in C for $(c_i, c_{i,(i+1)}, c_{i+1})$. Figure 3 illustrates an example of C and C'.

Since C is a cycle, every vertex in C is distinct. Also, note that C' is an even closed trail and, by the hypothesis, it has three short chords with an odd spacing. Since every vertex in C' that is in C has the same index parity and the three short chords of C' have an odd spacing, one of the short chords in C' must be one of the arcs of C. Given that such chord exists in $A(D)$, it is symmetric in D' from Lemma 5. Therefore, C has a symmetric arc.

Since every cycle of D' has at least one symmetric arc, it follows from Theorem 4 that D' has a kernel. Hence, from Lemma 3, D has a 3-kernel. ■

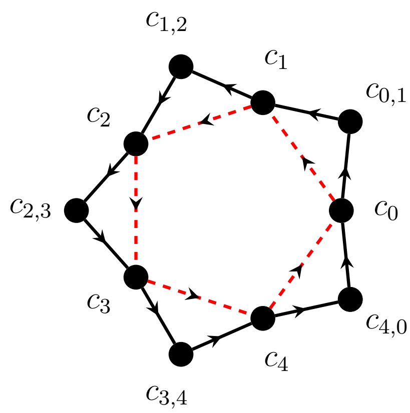

Figure 3. An illustration of a cycle C, in dashed red, and a closed trail C', in black.

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