# A new sufficient condition for the existence of 3-kernels 

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#### Abstract

Let $D$ be a digraph and $k$ be a positive integer. We say a subset $N$ of $V(D)$ is a $k$-kernel of $D$ if it is both $k$-independent and $(k-1)$-absorbent. A short chord of a closed trail $C=\left(v_{0}, v_{1}, \ldots, v_{t}\right)$ is an arc $a=\left(v_{i}, v_{j}\right)$ which does not belong to $C$ and the distance from $v_{i}$ to $v_{j}$ in $C$ is exactly two. The spacing between two chords $e=(u, v)$ and $f=(x, y)$ in $C$ is the distance from $u$ to $x$ in $C$. A set of chords in a closed trail $C$ has an odd spacing if at least two chords have an odd spacing. In this work, we prove that if $D$ is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then D has a 3-kernel.


## 1. Introduction

We assume that all digraphs have no loops or multiple arcs. The vertex set of a digraph $D$ is denoted by $V(D)$ and its arc set by $A(D)$. All trails, paths and cycles are considered to be directed. An arc $(u, v) \in A(D)$ is symmetric if $(v, u) \in A(D)$. A digraph $D$ is strongly connected if for every pair of vertices $u, v \in V(D)$, there exists a path from $u$ to $v$ in $D$. The distance between two vertices $u$ and $v$ in a digraph $D$, denoted by $d_{D}(u, v)$, is the length of the shortest path from $u$ to $v$ in $D$. We say that a closed trail or a cycle is even (resp. odd) if its length is even (resp. odd). For undefined notation, we refer the reader to [Bondy and Murty 2008].

A chord of a closed trail $C=\left(c_{0}, \ldots, c_{n-1}, c_{0}\right)$ is an arc $a=\left(c_{i}, c_{j}\right)$, where $c_{i}, c_{j} \in V(C)$ but $a \notin A(C)$. If the distance from $c_{i}$ to $c_{j}$ in $C$ is two, we say that it is a short chord. The spacing between two chords $e=(u, v)$ and $f=(x, y)$ of $C$ is the distance from $u$ to $x$ in $C$. A set of chords in a closed trail $C$ has an odd spacing if at least two chords have an odd spacing. Figure 1 illustrates a cycle with two odd spaced short chords.


Figure 1. A cycle with two odd spaced short chords.


Figure 2. An example of a digraph $D$ and its 2-closure $C^{2}(D)$. Those arcs in $C^{2}(D)$ which are not in $D$ are dashed and painted red.

A kernel of a digraph $D$ is a set $K \subseteq V(D)$ which is independent in $D$ and for every vertex $u \in V(D) \backslash K$, there exists an arc $(u, v)$, where $v \in K$. A digraph is kernelperfect if every induced subdigraph has a kernel. A subset $N$ of $V(D)$ is $k$-independent if for every pair of vertices $u, v \in N$, the distance from $u$ to $v$ in $D$ is at least $k$; also, $N$ is $\ell$-absorbent if for every vertex $u \in V(D) \backslash N$, there exists $v \in N$ such that the distance from $u$ to $v$ is at most $\ell$. We say that a subset $N$ of $V(D)$ is a $(k, \ell)$-kernel of $D$ if it is both $k$-independent and $\ell$-absorbent. A $k$-kernel is a $(k, k-1)$-kernel and a kernel is a 2-kernel.

The concept of kernel was introduced by von Neumann and Morgenstern in 1944 von Neumann and Morgenstern 1944] in the context of game theory to model social and economic interactions. Richardson [Richardson 1946] proved that every digraph which has no odd cycle has a kernel, a seminal result of the field. In light of Richardson's Theorem, kernel theory gained a lot of attention and was thoroughly researched, yielding new results regarding the existence of kernels in digraphs with odd cycles. In 1980, Kwasnik [Kwaśnik 1980] proposed the concept of $k$-kernels and later generalized Richardson's Theorem to $k$-kernels [Kwaśnik 1981].

In this work, we prove the following.
Theorem 1. Let $D$ be a digraph. If $D$ is a strongly connected digraph where every odd cycle has a short chord and every even closed trail has three short chords with an odd spacing, then D has a 3-kernel.

## 2. A sufficient condition for the existence of 3-kernels in digraphs

In this section, we prove Theorem 1. An useful tool for demonstrating the existence of 3 -kernels in digraphs is Lemma 3, which states that a digraph has a 3-kernel if, and only if, its 2 -closure has a kernel. Before we present Lemma 3, we introduce the definition of the $k$-closure of a digraph.
Definition 2. Let $D$ be a digraph. The $k$-closure of $D$, denoted by $C^{k}(D)$, is the digraph $D^{\prime}$, where $V\left(D^{\prime}\right)=V(D)$ and $(u, v) \in A\left(D^{\prime}\right)$ if $d_{D}(u, v) \leq k$. Figure 2 illustrates an example of a digraph and its 2-closure.
Lemma 3 ([Galeana-Sánchez and Hernández-Cruz 2014]). Let $k \geq 3$ be an integer. Let $D$ be a digraph and let $K \subseteq V(D)$. The subset of vertices $K$ is a $k$-kernel of $D$ if, and only if, $K$ is a kernel of $C^{(k-1)}(D)$.

A particularly useful theorem was proved by Duchet in 1980.

Theorem 4 ([Duchet 1980] $)$. If every cycle of a digraph $D$ has a symmetric arc, then $D$ is kernel-perfect.

The strategy of the proof to our theorem is to show that the 2 -closure of a digraph $D$ that satisfies the hypothesis from Theorem 1 has a symmetric arc in every cycle. From Theorem 4, the 2-closure of $D$ has a kernel; therefore, by Lemma 3 the digraph has a 3-kernel.

Before we present the demonstration, we must introduce the main lemma used in the proof.
Lemma 5. Let D be a strongly connected digraph. If every odd cycle in $D$ has a short chord and every even closed trail in $D$ has three short chords with an odd spacing, then for every $(u, v) \in A(D)$ there is a path of length at most two from $v$ to $u$ in $D$.

Proof. Let $f=(u, v) \in A(D)$. Since $D$ is strongly connected, there is a minimal path $T=\left(t_{0}=v, \ldots, t_{s}=u\right)$ from $v$ to $u$ in $D$. Note that $C=T \cup(u, v)$ is a cycle. Clearly, if $|C|=3$, then $|T|=2$ and the result follows. Assume, for the sake of contradiction, that $|T|>2$ and $|C|$ is even. Since $T$ is minimal, the only possible short chords in $C$ are $\left(t_{s-1}, v\right)$ and $\left(u, t_{1}\right)$, which contradicts the hypothesis of the existence of three short chords in $C$. Assume then, for the sake of contradiction, that $|T|>2$ and $|C|$ is odd. Due to the hypothesis and the minimality of $T, C$ has a short chord: $\left(t_{s-1}, v\right)$ or $\left(u, t_{1}\right)$. Let $(a, b)$ be one of such chords. Note that $(a, b) \cup(b, T, a)$ is a cycle of even length and, therefore, must have three short chords. Analogously to the former case, one of the chords connects two vertices in $T$, a contradiction. Hence, $|T| \leq 2$.

We are now ready to demonstrate Theorem 1 .
Proof of Theorem 1. Let $D^{\prime}=C^{2}(D)$. It follows from Lemma 3 that $D$ has a 3 -kernel if $D^{\prime}$ has a kernel. We show that every cycle in $D^{\prime}$ has a symmetric arc and, by Theorem 4 , $D^{\prime}$ has a kernel.

Assume that there exists a cycle $C=\left(c_{0}, c_{1}, \ldots, c_{n-1}, c_{0}\right)$ in $D^{\prime}$ with no symmetric arc. Choose such cycle $C$ with the shortest length in $D^{\prime}$. Note that no arc in $C$ exists in $D$, because it would be symmetric: from Lemma 5, if $\left(c_{i}, c_{i+1}\right) \in A(D)$, then $\left(c_{i+1}, c_{i}\right) \in A\left(D^{\prime}\right)$. Therefore, $d_{D}\left(c_{i}, c_{i+1}\right)=2$, for every $i \leq n$ (notation modulo $n$ ).

Since every arc $\left(c_{i}, c_{i+1}\right)$ of $C$ is in $A\left(D^{\prime}\right)$ - but not in $A(D)$ - for every arc $\left(c_{i}, c_{i+1}\right)$ in $C$ there exists a vertex $c_{i,(i+1)} \in V(D)$ such that $\left(c_{i}, c_{i,(i+1)}, c_{i+1}\right)$ exists in $D$. Let $C^{\prime}$ be the closed trail resulting from the substitution of every $\left(c_{i}, c_{i+1}\right)$ in $C$ for $\left(c_{i}, c_{i,(i+1)}, c_{i+1}\right)$. Figure 3 illustrates an example of $C$ and $C^{\prime}$.

Since $C$ is a cycle, every vertex in $C$ is distinct. Also, note that $C^{\prime}$ is an even closed trail and, by the hypothesis, it has three short chords with an odd spacing. Since every vertex in $C^{\prime}$ that is in $C$ has the same index parity and the three short chords of $C^{\prime}$ have an odd spacing, one of the short chords in $C^{\prime}$ must be one of the arcs of $C$. Given that such chord exists in $A(D)$, it is symmetric in $D^{\prime}$ from Lemma 5 . Therefore, $C$ has a symmetric arc.

Since every cycle of $D^{\prime}$ has at least one symmetric arc, it follows from Theorem 4 that $D^{\prime}$ has a kernel. Hence, from Lemma 3, $D$ has a 3 -kernel.


Figure 3. An illustration of a cycle $C$, in dashed red, and a closed trail $C^{\prime}$, in black.

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