Characterizing Networks Admitting k Arc-disjoint Branching Flows

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Abstract. *An s-branching flow f in a network $\mathcal{N} = (D, c)$ (where c is the capacity function) is a flow that reaches every vertex in $V(D) \setminus \{s\}$ from s while loosing exactly one unit of flow in each vertex other than s. In other words, the difference between the flow entering a vertex v and a flow leaving a vertex v is one whenever $v \neq s$. It is known that the hardness of the problem of finding k arc-disjoint s-branching flows in network $\mathcal{N}$ is linked to the capacity c of the arcs in $\mathcal{N}$: the problem is solvable in polynomial time if every arc has capacity $n - \ell$, for fixed ℓ, and* NP*-complete in most other cases, with very few cases open. We further investigate a conjecture by [Costa et al. 2019] that aims to characterize networks admitting k arc-disjoint s-branching flows, generalizing a result that provides such characterization when all arcs have capacity $n - 1$, based on Edmonds' branching theorem. We show that, in general, the conjecture is false. However, it holds for out-branchings with parallel arcs.*

1. Introduction

A *network* $\mathcal{N}$ is formed by a digraph D together with a *capacity function* $c : A(D) \to \mathbb{Z}_+$, and a *flow on* $\mathcal{N}$ is a function $f : A(D) \to \mathbb{Z}_+$ such that $f(e) \leq c(e)$ for all $e \in A(D)$. If e is an arc of D from a vertex u to a vertex v, we say that u and v are the *endpoints* of e and, for $v \in V(D)$, we denote by $E^-_D(v)$ and $E^+_D(v)$ the set of arcs reaching and leaving v in D, respectively. If all arcs in D have capacity λ, we simply write $c \equiv \lambda$. We denote by $f^+(v)$ and $f^-(v)$ the amount of flow leaving and entering v, respectively. That is, $f^+(v) = \sum_{e \in E^+_D(v)} f(e)$ and $f^-(v) = \sum_{e \in E^-_D(v)} f(e)$. Finally, we define the *balance* of a vertex v with respect to f as $f^+(v) - f^-(v)$ and denote it by $\mathsf{bal}(v)$.

In the classical (s,t)-Maximum Flow problem, the goal is to find a flow f maximizing $f^-(t)$ on the input network $\mathcal{N} = (D, c)$ such that $\mathsf{bal}(v) = 0$ for all $v \in V(D) \setminus \{s, t\}$, $f^-(s) = 0$, and $f^+(t) = 0$. A classical result by [Ford and Fulkerson 1956] shows that this problem is solvable in polynomial time. In [Bang-Jensen and Bessy 2014] many types of problems revolving around arc-disjoint flows are introduced and studied, generalizing well-known and important problems and showing that, in some cases, a polynomial time algorithm is possible. In this article, we consider one of such problems whose hardness is intrinsically connected to the capacity function of the network.

An s*-branching flow* on a network $\mathcal{N} = (D, c)$ is a flow f such that $\mathsf{bal}(s) = f^+(s) = n - 1$ and $\mathsf{bal}(v) = -1$ for all $v \in V(D) \setminus \{s\}$. In other words, f reaches all vertices in D, and each vertex other than s "consumes" one unit of flow.

For a digraph D and $X \subseteq V(D)$, let $d_D^-(X)$ be number of arcs from $V(D) \setminus X$ to X. We say that D is an *out-branching with root* s if there is path from s to every other vertex in D and the underlying graph of D is a tree.

A classical result by [Edmonds 1973] shows that the problem of finding k arc-disjoint out-branchings with root s is solvable in polynomial time. It was proved in [Bang-Jensen and Bessy 2014] that this also yields a polynomial time algorithm for the problem of finding k-arc disjoint s-branching flows in a given network $\mathcal{N} = (D, c)$ (henceforth abbreviated (s, k)-DBF) whenever $c \equiv n - 1$, and that this problem is NP-complete if every capacity at most 2, even if $k = 2$. The authors of [Bang-Jensen et al. 2016] extended this latter result by showing that, for fixed integer ℓ, (s, k)-DBF remains NP-complete if $c \equiv \ell$, but it is solvable in polynomial time if $c \equiv n - \ell$. Furthermore, they showed that, unless the *Exponential Time Hypothesis* (ETH - see [Impagliazzo et al. 2001]) fails, there is no polynomial time algorithm for (s, k)-DBF when $c \equiv \lambda$ for any choice of λ such that $n/2 \leq \lambda \leq n - (\log n)^{1+\varepsilon}$. The same holds if $(\log n)^{1+\varepsilon} \leq \lambda \leq n/2$, as showed in [Costa et al. 2019]. Figure 1 shows the currently known hardness results for (s, k)-DBF for different choices of the capacity function.

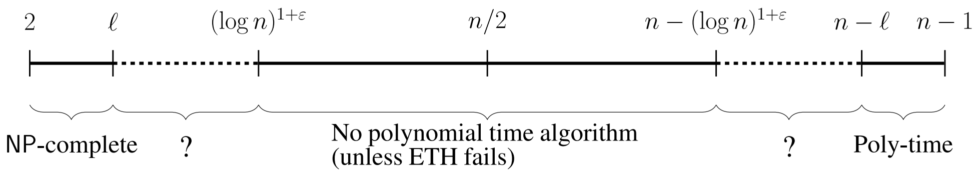

Figure 1. Complexity of (s, k)-DBF with regard to the capacity function.

The result in [Bang-Jensen and Bessy 2014] and [Edmonds 1973] gives a characterization of positive instances of (s, k)-DBF when the capacity is not an issue: it is shown that a network $\mathcal{N} = (D, c)$ with capacity $c \equiv n - 1$ admits k arc-disjoint s-branching flows if and only if $d^-(X) \geq k$ for all $X \subseteq V(D) \setminus \{s\}$. A generalization of this condition was proposed by [Costa et al. 2019]. Namely, the authors conjectured that a network $\mathcal{N} = (D, c)$ with $c \equiv \lambda$ admits k arc-disjoint s-branching flows if and only if

$$d_D^-(X) \geq k \cdot \left\lceil \frac{|X|}{\lambda} \right\rceil, \forall X \subseteq V(D) \setminus \{s\}. \qquad \text{(Property 1)}$$

The authors showed that this condition is not only necessary for any choice of λ but also sufficient for some particular choices of k and λ. It is also sufficient when D is a *multipath*, that is, D is a path if we ignore parallel arcs. We give a counterexample for this conjecture by showing a network whose digraph satisfies Property 1 without containing k arc-disjoint s-branching flows. On the positive side, we show that the conjecture holds for networks built on *multi out-branchings*; that is, D is an out-branching with root s with (possibly) parallel arcs. Since a multipath is also a multi out-branching, this generalizes the result of [Costa et al. 2019] mentioned above.

2. Arc-disjoint branching flows on networks satisfying Property 1

For a digraph D and $X \subseteq V(D)$, $D[X]$ denotes the subgraph of D *induced* by X. The *diameter* of D is the size of the largest shortest directed path between two vertices in D.

Lemma 2.1. *Let λ be a non-negative integer and D be a digraph such that Property 1 holds with respect to λ and some vertex $s \in V(D)$. Let $W \subseteq V(D) \setminus \{s\}$ be such that only one vertex $w \in W$ has in-neighbors in $V(D) \setminus W$. Then Property 1 holds for $D[W]$ with respect to λ and w.*

Proof. By our choice of w, we have $d^-_{D[W]}(v) = d^-_D(v)$, $\forall v \in W \setminus \{w\}$. Thus, for every $X \subseteq W \setminus \{w\}$ we have $d^-_{D[W]}(X) = d^-_D(X) \geq k \cdot \lceil |X|/\lambda \rceil$ and the result follows. $\square$

Theorem 2.2. *Let $\mathcal{N} = (D, c)$ be the network where D is a multi out-branching with root s and $c \equiv \lambda$. If Property 1 holds for D with respect to λ and s then $\mathcal{N}$ admits k arc-disjoint s-branching flows.*

Proof. We proceed by induction on the diameter h of D. If $h = 1$, then there are k arcs from s to every other vertex in $V(D)$ and the result follows. Assume now that the result holds for multi out-branchings with diameter $h-1$, and let $r_1, \ldots, r_p$ be the out-neighbors of s in D. Finally, for $i \in \{1, \ldots, p\}$, let D_i be the subgraph of D induced by the vertices that can be reached from r_i in D.

Now, by Lemma 2.1, Property 1 holds for each D_i with respect to r_i and λ. Furthermore, since the diameter of each D_i is at most $h - 1$, the network $\mathcal{N}_i = (D_i, c)$ admits k arc-disjoint r_i-branching flows $f'_1, \ldots, f'_k$. It remains to show how to extend each flow f'_i to an s-branching flow in $\mathcal{N}$. For $i \in \{1, \ldots, p\}$, let $q_i = |V(D_i)|/\lambda$ and $\alpha_i = |V(D_i)| - \lfloor q_i \rfloor \cdot \lambda$. For every $j \in \{1, \ldots, k\}$, start with $f_j(e) = f'_j(e)$ for every $e \in A(D)$ not containing s as an endpoint.

Now, for every $i \in \{1, \ldots, p\}$, partition the set of arcs from s to r_i into sets $E^i_1, \ldots, E^i_k$ of size $\lceil q_i \rceil$. Since Property 1 holds for D with respect to λ and s, we know that such partition is possible. Finally, add to the arcs used by each f_j exactly $\lfloor q_i \rfloor$ arcs from E^i_j with λ flow units each and one arc from the same set with α_i flow units, if $\alpha_i > 0$. Thus each f_j sends from s to r_i exactly $|V(D_i)|$ flow units and the result follows. $\square$

Figure 2 shows a counterexample for the conjecture in [Costa et al. 2019].

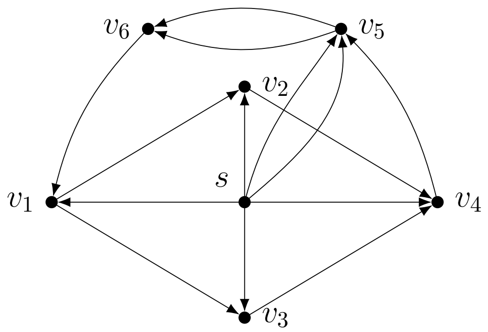

Figure 2. Counterexample for $\lambda = 2$ and $k = 2$.

Theorem 2.3. *Let D be the digraph shown in Figure 2 and $\mathcal{N} = (D, c)$ be a network with $c \equiv 2$. Then Property 1 holds for D with respect to $\lambda = 2$, s, and $k = 2$ and there are no 2 arc-disjoint s-branching flows in $\mathcal{N}$.*

Proof. We say that a set $X \subseteq V(D)$ is *feasible* if $d^-_D(X) \geq 2 \cdot \lceil |X|/2 \rceil$ and denote by $e(s, X)$ the number of arcs from s to the vertices in X. First, we show that every $X \subseteq V(D) \setminus \{s\}$ is feasible. Since $V(D) \setminus \{s\}$ is feasible, we assume that $X \subsetneq V(D) \setminus \{s\}$. We consider three cases.

If $v_6 \notin X$, then $e(s, X) \geq |X|$ which in turn implies that $d^-_D(X) \geq |X| + 1$ since $d^+_D(V(D) \setminus (X \cup \{s\})) \geq 1$ for all $X \subsetneq V(D) \setminus \{s\}$, and thus X is feasible. Similarly, if $v_6 \in X$ and $v_5 \in X$, we get that $d^-_D(X) \geq |X| + 1$ since there are two arcs from s to v_5 and thus X is feasible. Finally, if $v_6 \in X$ and $v_5 \notin X$, then $d^-_D(X) \geq e(s, X) + 2 \geq |X| - 1 + 2 = |X| + 1$ as there are two arcs from v_5 to v_6 in D. Thus we conclude that every $X \subseteq V(D) \setminus \{s\}$ is feasible.

Assume the contrary, i.e., that there are two arc-disjoint s-branching flows f_1 and f_2 in $\mathcal{N}$. Since there are only two arcs entering v_1 in D, one of them must be used by f_1 and the other by f_2, and the same holds for v_2 and v_3. Assume, without loss of generality, that the arc e from v_6 to v_1 is used by f_1 and that the arc e' from s to v_1 is used by f_2. If $f_1(e) = 1$ then either $f^-_1(v_2) = 0$ or $f^-_2(v_3) = 0$, a contradiction since both flows must reach all vertices in $V(D) \setminus \{s\}$. Thus, we know that $f_1(e) = 2$ which in turn implies that $f^-_1(v_6) = f^+_1(v_6) + 1 = 3$. Since each arc has capacity 2, we conclude that both arcs from v_5 to v_6 must be used by f_1 and thus $f^-_2(v_6) = 0$, contradicting our choice for f_2. $\square$

We observe that from the graph of Figure 2, a counterexample can be built for any even k and $\lambda \geq 2$. Although we have shown that Property 1 is not by itself sufficient for a network to have k arc-disjoint branching flows. We believe that combining it with some condition that take into account single vertices demands in the set X will lead to the sufficiency of the conjecture.

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