

The Rainbow Connection Number of Triangular Snake Graphs

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Abstract. *Rainbow coloring problems, of noteworthy applications in Information Security, have been receiving much attention last years in Combinatorics. The rainbow connection number of a graph G is the least number of colors for a (not necessarily proper) edge coloring of G such that between any pair of vertices there is a path whose edge colors are all distinct. In this paper we determine the rainbow connection number of the triple triangular snake graphs.*

1. Introduction

We address the problem of determining the *rainbow connection number* of a graph, introduced more than 10 years ago [Chartrand et al. 2008]. Rainbow colorings appear in many different contexts of Combinatorics [Ehard et al. 2019, Montgomery et al. 2019]. Let k be a non-negative integer and $c: E(G) \rightarrow \{0, 1, \dots, k-1\}$ a not necessarily proper edge-coloring of a graph G . A path P in G is *rainbow* if the colors of the edges in P are pairwise distinct. The edge-coloring c is a *rainbow coloring* if for every pair of vertices there is a rainbow path connecting them [Chartrand et al. 2008]. The *rainbow connection number* of G , denoted $rc(G)$, is the least number of colors for which there is a rainbow coloring of G . Rainbow connectivity has been receiving attention in the last years, being some works funded by the National Security Agency (NSA) of the United States [Lo 2012, Dudek et al. 2015], due to the applications on secure transferring of classified information after the terrorist attacks of September 11, 2001 [Li and Sun 2012].

In this paper, every graph is simple, connected, and non-trivial. Given a graph G , determining if $rc(G) \leq k$, for any fixed $k \geq 2$, is an NP-complete problem [Chakraborty et al. 2011, Ananth et al. 2011, Li and Li 2011]. In view of this hardness, several authors engaged efforts in the searching for upper and lower bounds for the rainbow connection number [Chakraborty et al. 2011, Chandran and Rajendraprasad 2012, Schiermeyer 2009]. Despite such effort, the problem is completely solved only for specific and structured graph classes, such as complete graphs, paths, and cycles [Chartrand et al. 2008].

In this paper we determine the rainbow connection number of the triple triangular snake graphs¹, a class commonly studied in labeling problems [Gallian 2014, Sandhya et al. 2016]. To the best of our knowledge, this parameter is unknown for this graphs, regardless of some investigation².

¹In Portuguese, *cobras triangulares triplas*.

²Rainbow connectivity has been considered for triangular snake graphs in a paper by D. Parmar, (cont.)

2. Preliminaries

The *distance* between two vertices u and v , denoted $dist(u, v)$, of a graph G is the number of edges in a shortest path between u and v in G . The *diameter* of G , denoted $diam(G)$, is the largest distance between a pair of vertices of G . Observe that, in a rainbow coloring of G , between two vertices u and v such that $dist(u, v) = diam(G)$ there must be a path requiring at least $diam(G)$ colors. This implies $rc(G) \geq diam(G)$ for any graph G . Moreover, $rc(G) \leq |V(G)| - 1$, since a rainbow coloring of G can be obtained assigning to the edges of a spanning tree of G colors which are all distinct [Chartrand et al. 2008].

Among the earliest results on rainbow coloring graph classes are: $rc(T_n) = n - 1$ for any tree T_n on n vertices; $rc(K_n) = 1$ for the complete graph on n vertices; $rc(C_n) = \lceil n/2 \rceil$ for the cycle C_n on n vertices [Chartrand et al. 2008]. Also, rainbow coloring has been receiving attention in the context of graph operations, specially the Cartesian product of graphs [Li and Sun 2010, Rocha and Almeida 2017].

Let ℓ and n be positive integers and $n \geq 2$. The ℓ -triangular snake graph, denoted TS_n^ℓ , is defined by $V(TS_n^\ell) = \{v_i : 0 \leq i < n\} \cup \{v_{i,i+1}^k : 0 \leq i \leq n-2 \wedge 0 \leq k < \ell\}$ and $E(TS_n^\ell) = \{v_i v_{i+1} : 0 \leq i \leq n-2\} \cup \{v_i v_{i,i+1}^k : 0 \leq i \leq n-2 \wedge 0 \leq k < \ell\} \cup \{v_{i,i+1}^k v_{i+1} : 0 \leq i \leq n-2 \wedge 0 \leq k < \ell\}$. If $\ell = 1$, the graph is simply called a *triangular snake*. If $\ell = 2$, it is called a *double triangular snake*. If $\ell = 3$, it is a *triple triangular snake* (see figures in the next section). The diameter of TS_n^ℓ is the distance between a vertex of $\{v_0\} \cup \{v_{0,1}^k : 0 \leq k < \ell\}$ and a vertex of $\{v_{n-1}\} \cup \{v_{n-2,n-1}^k : 0 \leq k < \ell\}$. When $n = 2$ and $\ell \geq 2$, the distance between v_0 and v_{n-1} is 2. When $n > 2$, the distance between v_0 and v_{n-1} is $n - 1$. So, if $n = 2$ and $\ell \geq 2$, then $diam(TS_n^\ell) = 2$; otherwise, $diam(TS_n^\ell) = n - 1$.

3. The result

We prove the following.

Theorem 1. *Let G be a triple triangular snake graph TS_n^3 .*

$$rc(G) = \begin{cases} n, & \text{if } n = 2 \text{ or } n = 3; \\ n - 1, & \text{if } n > 3. \end{cases}$$

Proof. Consider $n = 2$. It is sufficient to present a rainbow coloring for the graph G using $diam(G) = 2$ colors. Assign the color 1 for the edges $v_0 v_{0,1}^0$ and $v_{0,1}^2 v_1$. Assign the color 0 for the remaining edges. Clearly, this coloring is a rainbow coloring of G .

Consider $n = 3$. We shall first show that it is possible to construct a rainbow coloring for G using $diam(G) + 1 = 3$ colors. For all $k \in \{0, 1, 2\}$, assign the color 0 for the edges $v_0 v_1$ and $v_{0,1}^k v_1$, the color 1 for $v_1 v_2$ and $v_1 v_{1,2}^k$, and the color k for $v_0 v_{0,1}^k$ and $v_{1,2}^k v_2$. The rainbowness of this coloring is again straightforward to verify.

Still in the case $n = 3$, assume for the sake of contradiction that G has a rainbow coloring with only 2 colors. Let $X_i = \{v_i, v_{i+1}\} \cup \{v_{i,i+1}^k : 0 \leq k < 3\}$ for $i \in \{0, 1\}$.

(cont.) P. V. Shah and B. Suthar of 2019, entitled *Rainbow Connection Number of Triangular Snake Graph*, in a predatory journal. The authors correctly verify that the rainbow connection number of triangular snakes and double triangular snakes is equal to its diameter. For triple triangular snakes, we have found both the statement and the proof of the results wrong (the value claimed for the rainbow connection number could be greater than the diameter even for $n \geq 3$).

Without loss of generality, let 0 be the color of v_0v_1 . Since there is a rainbow path between v_0 and x for all $x \in X_1 \setminus X_0$, and since the 0-colored edge v_0v_1 is the first edge in these paths, the edges v_1v_2 and $v_1v_{1,2}^k$ for $0 \leq k < 3$ must be colored 1 (see Fig. 1(a)). This scenario leaves each among $v_{1,2}^0v_2v_{1,2}^1$ and $v_{1,2}^0v_2v_{1,2}^2$ as the only possibility for a rainbow path of length 2 between $v_{1,2}^0$ and each among $v_{1,2}^1$ and $v_{1,2}^2$, respectively. Since both paths share the edge $v_{1,2}^0v_2$, we must have $v_2v_{1,2}^1$ and $v_2v_{1,2}^2$ colored the same (see Fig. 1(b) for the case wherein $v_{1,2}^0v_2$ is colored 1). However, the only paths of length 2 between $v_{1,2}^1$ and $v_{1,2}^2$ are $v_{1,2}^1v_1v_{1,2}^2$ and $v_{1,2}^1v_2v_{1,2}^2$, which are both monochromatic, a contradiction.

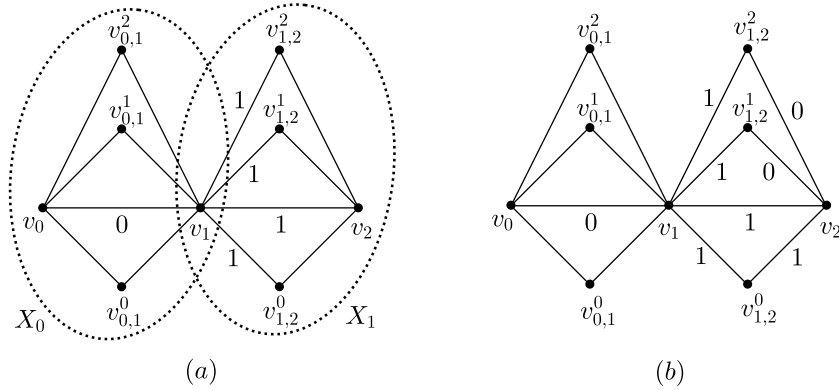


Figure 1. Constructing a rainbow coloring of TS_3^3

At last, consider $n > 3$. We shall construct a rainbow coloring for G using $\text{diam}(G) = n - 1$ colors. For $0 \leq i \leq n - 2$, the i^{th} block is defined as the subgraph induced by the edges $v_i v_{i+1}, v_i v_{i,i+1}^k, v_{i,i+1}^k v_{i+1}$ for all $k \in \{0, 1, 2\}$. The coloring of the first three blocks ($0 \leq i \leq 2$) is shown in Fig. 2.

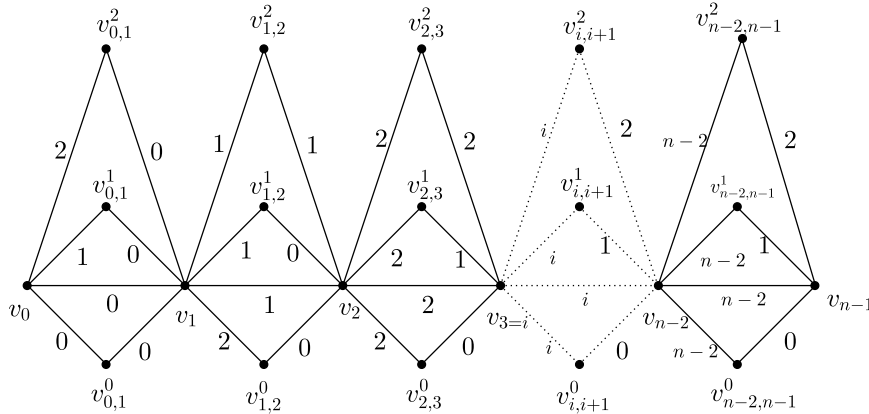


Figure 2. The rainbow coloring of the graph TS_n^3 with $n > 3$

For $i > 2$, the coloring of the i^{th} block generalizes the coloring for $i = 2$ in the following way: for $0 \leq k \leq 2$, the edges $v_i v_{i+1}$ and $v_i v_{i,i+1}^k$ are colored i , and the edges $v_{i,i+1}^k v_{i+1}$ are colored k (see again Fig. 2).

Now we verify that under the coloring shown there is a rainbow path between any pair of vertices u and v . If $u = v_{i,i+1}^{k'}$ and $v = v_{i,i+1}^{k''}$ are in the same i^{th} block for

$0 \leq i \leq n - 2$ and $\{k', k''\} \subseteq \{0, 1, 2\}$, then the rainbow path between them is the path of length 2 which passes through v_{i+1} . As it can also be verified in Fig. 2, in any other case there is only one shortest path between u and v , and this path is rainbow. \square

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