# New Bounds on Roller Coaster Permutations *

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***Abstract.*** *A* roller coaster *is a permutation $\pi$ that maximizes the sum $\mathrm{t}(\pi) = \sum_{\tau \in X(\pi)} \mathrm{id}(\tau)$, where $X(\pi)$ denotes the set of subsequences of $\pi$ with cardinality at least 3; and $\mathrm{id}(\tau)$ denotes the number of maximal increasing or decreasing subsequences of contiguous numbers of $\tau$. We denote by $\mathrm{t}_{\max}(n)$ the value $\mathrm{t}(\pi)$, where $\pi$ is a roller coaster of $\{1, \ldots, n\}$, for $n \geqslant 3$. Precise values of $\mathrm{t}_{\max}(n)$ for $n \leqslant 13$ were presented in [Ahmed and Snevily 2013]. In this paper, we explore the problem of computing lower bounds for $\mathrm{t}_{\max}(n)$. More specifically, we present a cubic algorithm to compute $\mathrm{t}(\pi)$ for any given permutation $\pi$; and an Integer Linear Programming model to obtain roller coasters. As a result, we improve known lower bounds found in the literature for $n \leqslant 40$.*

## 1. Introduction

Throughout the text, $[n]$ denotes the set $\{1, \ldots, n\}$. Fixed a positive integer $n$, a *permutation* $\pi$ of $[n]$ is an ordering $(\pi_1, \pi_2, \pi_3, ..., \pi_n)$ of $[n]$. We omit commas and parenthesis whenever doing so produces no ambiguity. The length of a permutation $\pi$ is the number of elements it contains, and we denote it by $|\pi|$. Let $S_n$ denote the set of all permutations of $[n]$, and let $|X|$ denote the cardinality of a set $X$. A *subsequence* $\tau$ of a permutation $\pi$, denoted by $\tau \subseteq \pi$, is a sequence obtained from $\pi$ by removing some (maybe none) of the elements of $\pi$, while keeping the order of the remaining elements.

Let $\mathrm{i}(\tau)$ (resp. $\mathrm{d}(\tau)$) be the number of maximal increasing (resp. decreasing) sequences of contiguous numbers in $\tau$, where a sequence of contiguous numbers consists of at least two consecutive numbers. Let $\mathrm{id}(\tau) = \mathrm{i}(\tau) + \mathrm{d}(\tau)$, and let $X(\pi)$ denote the set of every subsequence $\tau \subseteq \pi$ with at least three elements. Finally, we set $\mathrm{t}(\pi) = \sum_{\tau \in X(\pi)} \mathrm{id}(\tau)$. For illustration, we evaluate it on two permutations of $S_4$:

$$\begin{aligned}
\mathrm{t}(3412) &= \mathrm{id}(3412) + \mathrm{id}(341) + \mathrm{id}(342) + \mathrm{id}(312) + \mathrm{id}(412) \\
&= \quad 3 \quad + \quad 2 \quad + \quad 2 \quad + \quad 2 \quad + \quad 2 \quad = 11. \\
\mathrm{t}(1234) &= \mathrm{id}(1234) + \mathrm{id}(123) + \mathrm{id}(124) + \mathrm{id}(134) + \mathrm{id}(234) \\
&= \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad + \quad 1 \quad = 5.
\end{aligned}$$

We define $\mathrm{t}_{max}(n)$ as the maximum $\max_{\pi \in S_n} \mathrm{t}(\pi)$, and say that a permutation $\pi$ is a *roller coaster* if $\mathrm{t}(\pi) = \mathrm{t}_{max}(n)$. In the example above, 1234 is not a roller coaster because $\mathrm{t}(1234) < \mathrm{t}(3412)$. On the other hand, it can be verified that 3412 is a roller coaster by checking that $\mathrm{t}_{max}(4) = 11$. Finally, $RC(n)$ denotes the set of roller coasters of length $n$.

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Roller Coaster permutations were introduced in [Ahmed and Snevily 2013], where the authors present values for $\mathrm{t}_{max}(n)$ for $n \leqslant 13$, and a construction that provides lower bounds for $n \geqslant 14$. The definition of t suggests a $\theta(2^n)$ algorithm to compute t, and one would take $O(2^n n!)$ time exploring every permutation of length $n$ in order to obtain $\mathrm{t}_{\max}(n)$. In Section 2, we present an $O(n^3)$ algorithm to compute t; and in Section 3, an Integer Linear Programming model to find roller coasters, together with new bounds for $\mathrm{t}_{max}(n)$ for $n$ up to 40.

## 2. Fast Computation of $\mathrm{t}(\boldsymbol{\pi})$

Let $\pi = \pi_1\pi_2\pi_3...\pi_n \in S_n$ and let $1 < j < n$. We call the triple $\pi_{j-1}\pi_j\pi_{j+1}$ a *peak* (resp. *valley*) if $\pi_j > \pi_{j-1}, \pi_{j+1}$ (resp. $\pi_j < \pi_{j-1}, \pi_{j+1}$). Given a subsequence $\tau$ of $\pi$, we denote by $\mathrm{p}(\tau)$ (resp. $\mathrm{v}(\tau)$) the number of peaks (resp. valleys) of $\tau$. For example, $p(42135) = 0$ but $v(42135) = 1$. Given $1 \leqslant i < j < k \leqslant n$, denote by $\Delta(\pi_i, \pi_j, \pi_k)$ the sum $\mathrm{p}(\pi_i\pi_j\pi_k) + \mathrm{v}(\pi_i\pi_j\pi_k)$. Finally, we say that $\pi_i\pi_j\pi_k$ is a *triangle* if $\Delta(\pi_i, \pi_j, \pi_k) = 1$.

**Proposition 1.** *Let* $\tau \in S_n$. *Then* $\mathrm{id}(\tau) = 1 + \mathrm{p}(\tau) + \mathrm{v}(\tau)$.

*Proof.* Let $r = \mathrm{id}(\tau)$, and let $\tau_1, \tau_2, \ldots, \tau_r$ be the maximal increasing and decreasing contiguous subsequences of $\tau$, in the order that they appear in $\tau$, where $\tau_i = \tau_1^i \cdots \tau_{s_i}^i$. Fix $i \in \{2, \ldots, r\}$. Note that $\tau_{s_{i-1}}^{i-1} = \tau_1^i$. By the maximality of $\tau_i$, we have that $\tau_i$ is increasing if and only if $\tau_{i-1}$ is decreasing. This implies that $\tau_{s_{i-1}-1}^{i-1}\tau_1^i\tau_2^i$ is either a peak or a valley. Moreover, these are the only peaks and valleys of $\tau$. Therefore, $\mathrm{p}(\tau) + \mathrm{v}(\tau) = r - 1 = \mathrm{id}(\tau) - 1$ as desired. $\square$

The next result provides an $O(n^3)$ algorithm for computing t.

**Theorem 1.** *Let* $\pi \in S_n$. *Then*

$$\mathrm{t}(\pi) = |X(\pi)| \quad + \sum_{1 \leqslant i < j < k \leqslant n} 2^{n-(k-i+1)}\Delta(\pi_i, \pi_j, \pi_k). \tag{1}$$

*Proof.* First, by Proposition 1, we have:

$$\mathrm{t}(\pi) = \sum_{\tau \in X(\pi)} \mathrm{id}(\tau) = \sum_{\tau \in X(\pi)} (1 + \mathrm{p}(\tau) + \mathrm{v}(\tau)) = |X(\pi)| \quad + \sum_{\tau \in X(\pi)} (\mathrm{p}(\tau) + \mathrm{v}(\tau)).$$

Now, we double count the cardinality of the following set.

$$E = \{(\tau, \sigma) : \tau = \tau_1 \cdots \tau_r \in X(\pi), \sigma = \tau_i\tau_{i+1}\tau_{i+2}, 1 \leqslant i \leqslant r-2 \text{ such that } \Delta(\sigma) = 1\}.$$

Let $d_1(\tau)$ (resp. $d_2(\sigma)$) denote $|\{(x, y) \in E : x = \tau\}|$ (resp. $|\{(x, y) \in E : y = \sigma\}|$). Clearly $\sum_{\tau \in X(\pi)} d_1(\tau) = |E| = \sum_{\sigma \in X(\pi), |\sigma|=3} d_2(\sigma)$, and $d_1(\tau)$ is the number of triangles of $\tau$, i.e., $d_1(\tau) = \mathrm{p}(\tau) + \mathrm{v}(\tau)$. Note that, for any subsequence $\sigma = \pi_i\pi_j\pi_k$ of $\pi$ for which $\Delta(\sigma) = 1$, the pair $(w_1\sigma w_2, \sigma) \in E$, for every $w_1 \subseteq \pi_1...\pi_{i-1}$ and $w_2 \subseteq \pi_{k+1}...\pi_n$. Therefore, $d_2(\sigma) = 2^{n-(k-i+1)}$, which is the number of permutations $w_1\sigma w_2$. Consequently, $\sum_{\sigma \in X(\pi), |\sigma|=3} d_2(\sigma) = \sum_{i<j<k} \Delta(\pi_i, \pi_j, \pi_k) d_2(\pi_i\pi_j\pi_k) = \sum_{1 \leqslant i < j < k \leqslant n} 2^{n-(k-i+1)}\Delta(\pi_i, \pi_j, \pi_k)$. Therefore, we have:

$$\sum_{\tau \in X(\pi)} (\mathrm{p}(\tau) + \mathrm{v}(\tau)) = |E| = \sum_{1 \leqslant i < j < k \leqslant n} 2^{n-(k-i+1)}\Delta(\pi_i, \pi_j, \pi_k), \tag{2}$$

which leads to the desired result. $\square$

Let the value $b = k - i + 1$ be the *basis* of the triangle $\pi_i\pi_j\pi_k$, and let $\Delta_b(\pi)$ be the number of triangles with basis $b$ in $\pi$. Equation (1) can be rewritten as $\mathrm{t}(\pi) = \sum_{b=3}^{n} 2^{n-b}\Delta_b(\pi) + |X(\pi)|$, which shows a single triangle with a smaller basis contributes more than a single triangle with larger basis. This supports the following conjecture, posed in [Ahmed and Snevily 2013], and also its strengthening anounced in [Adamczak 2016], that says that, for every $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$, we have either $1 \leqslant \pi_{2k+2} \leqslant n/2$ and $n/2 \leqslant \pi_{2k+1} \leqslant n$ or $1 \leqslant \pi_{2k+1} \leqslant n/2$ and $n/2 \leqslant \pi_{2k} \leqslant n$.

**Conjecture 1** (Ahmed-Snevily, 2013). *If $\pi \in RC(n)$, then $\pi_{k+1} > \pi_k, \pi_{k+2}$ or $\pi_{k+1} < \pi_k, \pi_{k+2}$, for every $k \in \{1, \ldots, n-2\}$.*

## 3. An Integer Linear Programming Model

In this section we present an integer linear programming model to find roller coasters of a given size $n$. Its objective function is derived from Equation (1) and the main variable $\mathbf{x} = (x_1, x_2, ..., x_n)$ represents the permutation itself. We use auxiliary binary variables $p_{i,j,k}, v_{i,j,k}$ with $1 \leqslant i < j < k \leqslant n$, and $w_{i,j}$ with $1 \leqslant i < j \leqslant n$, where $p_{i,j,k}$ (resp. $v_{i,j,k}$) indicates whether $x_ix_jx_k$ is a peak (resp. a valley), and $w_{i,j}$ indicates whether $x_i > x_j$. A binary variable equals $1$ if its property is satisfied, and $0$ otherwise.

For $(x_1, \ldots, x_n)$ to be a permutation, we must have $x_i \neq x_j$, for every $i \neq j$, which is expressed by Equations (3b) and (3c). For $x_ix_jx_k$ to be a triangle, $x_ix_jx_k$ must be either a peak, for which we have $x_j > x_i$ and $x_j > x_k$, and can be expressed by equations $x_j \geqslant x_i - n(1-p_{i,j,k}) + 1$ and $x_j \geqslant x_k - n(1-p_{i,j,k}) + 1$; or a valley, for which we have $x_j < x_i$ and $x_j < x_k$, and can be expressed by equations $x_j \leqslant x_k + n(1-v_{i,j,k}) - 1$ and $x_j \leqslant x_i + n(1-v_{i,j,k}) - 1$. These constraints are denoted by $PV_{i,j,k}$ (see Equation (3d)).

**Model 1. An Integer Programming Model for finding roller coasters.**

$$\max \quad \mathrm{t}(\mathbf{x}) = \sum_{1 \leqslant i < j < k \leqslant n} 2^{-(k-i+1)} \left(p_{i,j,k} + v_{i,j,k}\right) \tag{3a}$$

$$\text{s.t.} \quad w_{i,j} + w_{j,i} = 1, \qquad \forall\, i \neq j, \tag{3b}$$

$$x_i \geqslant x_j + n(w_{i,j} - 1) + 1, \qquad \forall\, i \neq j, \tag{3c}$$

$$PV_{i,j,k}, \qquad \forall\, i < j < k. \tag{3d}$$

Unfortunately, we were not able to run this model for $n \geqslant 18$. On the other hand, by using Adamczak's strengthening of Conjecture 1 as additional constraints, which are translated to $x_i \geqslant n/2$ when $i$ is even, and $x_i \leqslant n/2$ when $i$ is odd, we were able to obtain new permutations for $n$ up to $40$ (see Table 2). These new permutations improved some of the lower bounds on $\mathrm{t}_{\max}$ known so far (see Table 1). Note that if the strengthening of Conjecture 1 holds, then these additional constraints exclude only the solutions for which $x_i \leqslant n/2$ when $i$ is even, and $x_i \geqslant n/2$ when $i$ is odd, and hence do not exclude all optimal solutions, which implies that the permutations found are indeed roller coasters, and their respective values of $\mathrm{t}$ are $\mathrm{t}_{\max}$. Our experiments were written on Sagemath [The Sage Developers 2020] and ran with the Gurobi solver [Gurobi Optimization 2021].

## 4. Conclusion and Future work

This paper presents an alternative and fast algorithm to calculate $\mathrm{t}$, and an integer linear programming model to find roller coasters, which provided us with new lower bounds for $t_{max}$. We plan to explore Conjecture 1 in order to prove it or disprove it. While our methods have frequently supported the validity of Conjecture 1, by excluding the additional constraints, the model may be able to find a counterexample for it.

**Table 1. Permutations found using Model 1.**

| N | |
|---|---|
| 14 | [7, 11, 3, 13, 5, 9, 1, 14, 6, 10, 2, 12, 4, 8] |
| 15 | [7, 12, 3, 14, 5, 10, 1, 15, 6, 9, 2, 13, 4, 11, 8] |
| 16 | [8, 12, 4, 14, 2, 10, 6, 16, 1, 11, 7, 15, 3, 13, 5, 9] |
| 17 | [8, 14, 3, 15, 6, 10, 2, 17, 7, 12, 1, 16, 5, 11, 4, 13, 9] |
| 18 | [9, 14, 4, 16, 7, 11, 2, 18, 6, 13, 1, 17, 8, 12, 3, 15, 5, 10] |
| 19 | [9, 15, 5, 17, 2, 12, 8, 19, 3, 13, 6, 16, 1, 11, 7, 18, 4, 14, 10] |
| 20 | [10, 15, 5, 18, 2, 12, 8, 20, 4, 14, 7, 17, 1, 13, 9, 19, 3, 16, 6, 11] |
| 21 | [10, 17, 4, 19, 8, 13, 1, 21, 6, 15, 3, 18, 9, 12, 2, 20, 7, 14, 5, 16, 11] |
| 22 | [11, 17, 5, 20, 8, 14, 2, 22, 10, 16, 4, 19, 7, 13, 1, 21, 9, 15, 3, 18, 6, 12] |
| 23 | [11, 17, 6, 21, 3, 15, 9, 23, 1, 16, 7, 19, 4, 13, 10, 22, 2, 14, 8, 20, 5, 18, 12] |
| 24 | [12, 18, 6, 21, 3, 15, 9, 23, 1, 17, 11, 20, 5, 14, 8, 24, 2, 16, 10, 22, 4, 19, 7, 13] |
| 25 | [12, 19, 5, 16, 9, 23, 2, 18, 10, 25, 1, 14, 7, 21, 4, 17, 11, 24, 3, 15, 8, 22, 6, 20, 13] |
| 26 | [13, 20, 6, 23, 10, 16, 3, 25, 8, 18, 1, 22, 12, 15, 5, 26, 9, 19, 2, 24, 11, 17, 4, 21, 7, 14] |
| 27 | [13, 20, 7, 24, 3, 17, 11, 23, 5, 19, 9, 27, 1, 15, 12, 22, 4, 18, 8, 26, 2, 16, 10, 25, 6, 21, 14] |
| 28 | [14, 21, 7, 25, 3, 17, 11, 27, 5, 19, 9, 23, 1, 16, 13, 28, 6, 20, 10, 24, 2, 18, 12, 26, 4, 22, 8, 15] |
| 29 | [14, 23, 7, 26, 11, 18, 3, 28, 9, 20, 5, 24, 13, 16, 1, 29, 8, 21, 4, 25, 12, 17, 2, 27, 10, 19, 6, 22, 15] |
| 30 | [15, 23, 7, 27, 11, 19, 3, 29, 13, 21, 5, 25, 9, 17, 1, 30, 14, 22, 6, 26, 10, 18, 2, 28, 12, 20, 4, 24, 8, 16] |
| 31 | [15, 24, 8, 27, 11, 20, 3, 29, 13, 18, 5, 25, 9, 22, 1, 31, 14, 17, 6, 26, 10, 21, 2, 30, 12, 19, 4, 28, 7, 23, 16] |
| 32 | [16, 24, 8, 28, 4, 20, 12, 30, 2, 22, 14, 26, 6, 18, 10, 32, 1, 23, 15, 27, 7, 19, 11, 31, 3, 21, 13, 29, 5, 25, 9, 17] |
| 33 | [16, 26, 8, 29, 11, 20, 3, 31, 13, 22, 5, 27, 10, 18, 1, 33, 15, 24, 6, 28, 9, 19, 2, 32, 14, 23, 4, 30, 12, 21, 7, 25, 17] |
| 34 | [17, 26, 8, 30, 13, 21, 4, 32, 11, 23, 2, 28, 15, 19, 6, 34, 10, 25, 1, 29, 16, 20, 7, 33, 12, 24, 3, 31, 14, 22, 5, 27, 9, 18] |
| 35 | [17, 27, 9, 31, 4, 22, 14, 29, 6, 23, 12, 34, 2, 20, 16, 30, 7, 25, 10, 35, 1, 19, 15, 28, 5, 24, 11, 33, 3, 21, 13, 32, 8, 26, 18] |
| 36 | [18, 27, 9, 32, 4, 22, 14, 34, 7, 24, 12, 29, 2, 20, 16, 36, 6, 26, 11, 31, 1, 21, 17, 35, 8, 25, 13, 30, 3, 23, 15, 33, 5, 28, 10, 19] |
| 37 | [18, 29, 9, 24, 14, 34, 3, 21, 11, 31, 6, 26, 17, 36, 2, 22, 12, 32, 7, 27, 15, 37, 1, 20, 10, 30, 5, 25, 16, 35, 4, 23, 13, 33, 8, 28, 19] |
| 38 | [19, 29, 9, 34, 14, 24, 5, 36, 17, 27, 7, 31, 11, 21, 2, 38, 16, 26, 6, 33, 13, 23, 1, 37, 18, 28, 8, 32, 12, 22, 3, 35, 15, 25, 4, 30, 10, 20] |
| 39 | [19, 30, 10, 35, 14, 24, 4, 38, 17, 22, 7, 31, 11, 27, 1, 37, 16, 23, 6, 33, 13, 28, 3, 39, 18, 21, 8, 32, 12, 26, 2, 36, 15, 25, 5, 34, 9, 29, 20] |
| 40 | [20, 30, 11, 35, 5, 25, 15, 38, 2, 27, 18, 32, 7, 22, 13, 40, 3, 29, 17, 34, 9, 24, 12, 37, 1, 28, 19, 33, 8, 23, 14, 39, 4, 26, 16, 36, 6, 31, 10, 21] |

**Table 2. Values of t(n) obtained by 'AS', as in [Ahmed and Snevily 2013], and with 'BN', as the integer linear programming model, for n = 14,. . . ,40. Improved lower bounds are presented with bold text.**

| | 14 | 15 | 16 | 17 | 18 |
|---|---|---|---|---|---|
| AS | 81350 | 174954 | 374409 | 798471 | 1700036 |
| BN | 81350 | 174954 | 374409 | **798783** | 1700036 |
| | 19 | 20 | 21 | 22 | 23 |
| AS | 3596124 | 7588303 | 15970785 | 33596706 | 70310126 |
| BN | **3597020** | 7588303 | 15970785 | 33596706 | 70310126 |
| | 24 | 25 | 26 | 27 | 28 |
| AS | 146867861 | 306492900 | 639129568 | 1327542841 | 2755084935 |
| BN | 146867861 | **306500899** | **639198976** | **1328781760** | **2758443963** |
| | 29 | 30 | 31 | 32 | 33 |
| AS | 5720021634 | 11863992638 | 24524469439 | 50593221917 | 104565405932 |
| BN | **5720153893** | 11863992638 | **24525731250** | **50650675297** | **104569114183** |
| | 34 | 35 | 36 | 37 | 38 |
| AS | 215826275292 | 444271587981 | 914139811651 | 1881877624386 | 3872524536090 |
| BN | **215844113148** | **444587412964** | **914999923559** | **1882036116393** | **3872525917922** |
| | 39 | 40 | | | |
| AS | 7948257224143 | 16292370258569 | | | |
| BN | **7949294221494** | **16308000242795** | | | |

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