On $L(h,k)$-labelings of oriented graphs *

Lucas Colucci

[1]Instituto de Matemática e Estatística
Universidade de São Paulo
Brazil
lcolucci@ime.usp.br

Abstract. *We compare the behaviour of the $L(h,k)$-number of undirected and oriented graphs in terms of maximum degree, highlighting differences between the two contexts. In particular, we prove that, for every h and k, oriented graphs with bounded degree in every block of their underlying graph (for instance, oriented trees and oriented cacti) have bounded $L(h,k)$-number, giving an upper bound on this number which is sharp up to a multiplicative factor* 4.

1. Introduction

Given a pair of integers (h,k), with $h \geq 0$ and $k \geq 1$, an $L(h,k)$-*labeling* or $L(h,k)$-*coloring* of a graph (without multiple edges or loops) G is an assignment $f : V(G) \to \{0,1,\ldots,t\}$ such that, for every u,$v \in V(G)$, we have $|f(u) - f(v)| \geq h$, if u and v are joined by an edge; and $|f(u) - f(v)| \geq k$ if there is a path of length two joining u and v. The minimum value of t among the $L(h,k)$ colorings of G is called the $L(h,k)$-*number* of G, and it is denoted by $\lambda_{h,k}(G)$. Note that a $L(h,k)$-coloring with $\lambda_{h,k}(G) = t$ can use $t+1$ colors (as 0 is included in the range of f). Although this is a slightly counterintuitive detail in the definition, we follow it for historical reasons. Also, contrarily to a good part of the literature, we do not assume in general that $h \geq k$ (even though this is the most relevant case in terms of applications). The case $k = 0$ is excluded since it is easy to see that $\lambda_{h,0}(G) = h(\chi(G) - 1)$, where $\chi(G)$ is the usual chromatic number of G, i.e., $L(h,0)$-labelings are essentially usual vertex colorings scaled by a constant factor.

At first, $L(h,k)$-labelings were introduced in [Yeh 1990] in the special case $h = 2$ and $k = 1$, and their origin traces back to the frequency assignment problem of wireless networks introduced in [Hale 1980], where a similar mathematical model was used to deal with the so-called Channel Assignment Problem.

The definition of $L(h,k)$-labelings extends naturally to oriented graphs (which are directed graphs whose underlying graph is simple), namely, if G is an oriented graph, an $L(h,k)$-labeling of G is a function $f : V(G) \to \{0,\ldots,t\}$ such that $|f(u) - f(v)| \geq h$, if u and v are joined by an edge; and $|f(u) - f(v)| \geq k$, if there is a *directed* path of length two joining u and v. The corresponding $L(h,k)$-number is usually denoted by $\overrightarrow{\lambda}_{h,k}(G)$. These labelings were first considered by Chang and Liaw [Chang and Liaw 2003] (for $(h,k) = (2,1)$), and in more generality in [Chang et al. 2007] and [Chen et al. 2009].

In this paper, we will compare the $L(h,k)$-number of undirected and oriented graphs, and prove that, while for undirected graphs, this parameter can be estimated from

*The author is supported by FAPESP (2020/08252-2). FAPESP is the São Paulo Research Foundation.

above and from below in terms of the maximum degree of the graph, in the case of oriented graphs, the relevant parameter to bound $\overrightarrow{\lambda}_{h,k}$ is the maximum of the in- and outdegrees inside each block (i.e., biconnected component, which in this paper might be a single edge as well) of the underlying graph.

2. Undirected versus oriented graphs

For every oriented graph G, it is clear that $\overrightarrow{\lambda}_{h,k}(G) \leq \lambda_{h,k}(H)$, where H is the underlying graph of G. However, this inequality is usually far from sharp. Indeed, these two parameters behave quite differently: for undirected graphs, there is a lower bound for $\lambda_{h,k}$ in terms of the maximum degree, namely $\lambda_{h,k}(H) \geq (\Delta(H) - 1)k$, as every pair of vertices in the neighborhood of a vertex of degree $\Delta(H)$ must have colors with difference at least k (this bound can be improved to $\lambda_{h,k}(H) \geq h + (\Delta(H) - 1)k$ if $h \geq k/2$). For oriented graphs, on the other hand, there is no such phenomenon, since the neighborhood of every vertex can be locally colored with two colors with difference at least k (one for the inneighbors and the other for the outneighbors). In fact, there is no lower bound for $\overrightarrow{\lambda}_{h,k}$ in terms of the maximum degree: for instance, the results in [Chang et al. 2007] imply that $\overrightarrow{\lambda}_{h,k}(T) \leq h + 3k$ for every oriented tree T.

Also, for an undirected graph G with maximum degree Δ, note that a greedy coloring shows that $\lambda_{h,k}(G) \leq \Delta(2h - 1) + (\Delta^2 - \Delta)(2k - 1)$. On the other hand, the incidence graph of a projective plane shows that there are graphs G with $\lambda_{h,k}(G) \geq k\Delta(G)^2 + c_{h,k} \cdot \Delta(G)$, for some $c_{h,k}$ which depends only on h and k. This shows that the upper bound is sharp up to a factor of 2.

Theorem 1 shows that the relevant parameter in the case of oriented graphs is actually the maximum degree inside a block of the corresponding underlying graph. Together with Theorem 2, it shows that bounds similar to the one in the paragraph above for undirected graphs hold for an oriented graph if we replace the global maximum degree by the maximum degree inside the blocks of the underlying graph. In particular, it implies that the $L(h, k)$-number of oriented trees, oriented cacti and oriented graphs with bounded number of cycles in their underlying graph is bounded from above by a constant that depends only on h and k.

Theorem 1. *Let $d \geq 1$ be an integer, h, k be integers with $h \geq 0$ and $k \geq 1$, and let G be an oriented graph with the following property: for every block B of its underlying graph, the in- and outdegrees of the vertices of $G[B]$, the subgraph of G induced by $V(B)$, are bounded by d. Then*

$$\overrightarrow{\lambda}_{h,k}(G) \leq \begin{cases} (4k-2)d^2 + (4h-2)d, & \text{if } h \geq 1; \\ (4k-2)d^2, & \text{if } h = 0. \end{cases}$$

Proof. We will prove the result only in the case $h \geq 1$, as the other case is analogous. We may assume that the underlying graph of G, which we call H, is connected (as the proof and the result applies in each component).

Put $M = \max\{h, k\}$. The proof goes by induction on the number of blocks of H. If H has only one block (that is, H is biconnected), it is possible to color the vertices of G greedily using at most $(4k-2)d^2 + (4h-2)d + 1$ colors. In fact, every vertex v of H has at most $2d$ neighbors, each of which forbids at most $2h-1$ colors for v, and at most $2d^2$ non-neighbors joined to v by a path of length two, each of which forbids at most $2k-1$ colors for v.

Suppose now that H contains at least two blocks. Let v be the neighbor of a terminal vertex in a longest path in the block-cut tree of H, which is the tree whose vertices are the blocks and the cutvertices of H, and where two vertices are joined if they correspond to a cutvertex and a block containing it. The vertex v has the following property: it is a cut vertex of H, and at most one block of H that contains v is not a leaf block (i.e., contain another cut vertex of H). Let $B_1, \ldots, B_t$ be the leaf blocks of H that contain v. By induction hypothesis, the oriented graph $G' = G - \bigcup_{i=1}^{t}(V(B_i)\setminus\{v\})$ has an $L(h,k)$-labeling with at most $(4k-2)d^2 + (4h-2)d + 1$ colors. We are left with the vertices of $\bigcup_{i=1}^{t}(V(B_i)\setminus\{v\})$ to color to extend this coloring to a $L(h,k)$-labeling of G.

Let X_i (resp. Y_i) denote the set of vertices in B_i that points to (resp. from) v in G, and put $X = \bigcup_{i=1}^{t} X_i$ and $Y = \bigcup_{i=1}^{t} Y_i$. The condition on the degrees inside a block implies that $|X_i| \leq d$ and $|Y_i| \leq d$ for every $1 \leq i \leq t$. Furthermore, for every $i \neq j$, there is no edge joining X_i and X_j, Y_i and Y_j, or X_i and Y_j. There are two cases:

Case 1: $d \geq 2$ or $h \geq k$. The vertex v has at most $2d$ neighbors in G', each of which forbids at most $2k-1$ colors for a vertex in $A \cup B$, and v itself forbids at most $2M-1$ colors for a vertex in $A \cup B$. This means that we have at least $(4k-2)d^2 + (4h-2)d + 1 - 2d(2k-1) - (2h-1)$ free colors to use in the vertices of $X \cup Y$. A straightforward calculation shows that this number is at least $C = (2d-1)M + 1$ whenever $d \geq 2$ or $h \geq k$. In these cases, we can use these free colors to color $X \cup Y$. Indeed, suppose $c_1 < \cdots < c_{(2d-1)M+1}$ are C of the free colors. We can take the d colors $c_1, c_{M+1}, c_{2M+1}, \cdots, c_{(d-1)M+1}$ to use in X and the d colors $c_{dM+1}, c_{(d+1)M+1}, c_{(d+2)M+1}, \cdots, c_{(2d-1)M+1}$ to use in Y: to color X, we need at most d colors (since $|X_i| \leq d$, and hence inside a component of X_i and Y_i, we may assign the colors to vertices without repetition) which are at least M apart from each other, and the same for Y. Moreover, the choice of colors above guarantees that every color used in X and in Y are at least M distant from each other. Finally, we have to color the vertices of $\bigcup_{i=1}^{t}(V(B_i)\setminus\{v\})$ which are not neighbors of v. These vertices can be colored greedily as in the base case of the induction, since all the neighbors and second-neighbors of such vertices lie inside a block, wherein the maximum degree is at most d.

Case 2: $d = 1$ and $k > h$. In this case, the induction step consists of a more subtle argument. As before, assume that G' is colored with at most $(4k-2)d^2 + (4h-2)d + 1 = 4(k+h) - 3$ colors (the integers from 0 to $4h + 4k - 4$), and call such a coloring c. As $d = 1$, v has at most one outneighbor x and one inneighbor y in G', and $|X_i| \leq 1$ and $|Y_i| \leq 1$ for every $1 \leq i \leq t$. The vertex v forbids an interval S_v of at most $2k-1$ colors to use in $X \cup Y$, and the vertex x (resp. y) forbids an interval S_x (resp. S_y) of at most $2k-1$ colors to use in X (resp. Y). Our goal is to find two colors, say α and β, such that $\alpha \notin S_x \cup S_v$, $\beta \notin S_y \cup S_v$ and $|\alpha - \beta| \geq k$, so we may assign α to the vertices in X, β to the vertices in Y and complete the induction step as before.

Assume first that $\{0, 4h+4k-2\} \cap S_v = \emptyset$. In this case, each of $0, 4h+4k-2$ do not belong to one of S_x, S_y (as $|S_x|, |S_y| \leq 2k-1 < 4k+4h-2$). Also, it is clear that S_x and S_y cannot both contain 0 (or $4h+4k-2$), for if S_x and S_y both contain one of these, then either $0 \leq c(x), c(y) \leq k-1$ or $4h+4k-2-(k-1) \leq c(x), c(y) \leq 4h+4k-2$, which contradicts $|c(x)-c(y)| \geq k$. This implies that there is a choice of α and β with $\{\alpha, \beta\} = \{0, 4h+4k-2\}$.

On the other hand, if, say, $0 \in S_v$ (the other case is symmetric), we know that $c(v) \leq k-1$, and hence $S_v \cap \{2k-1, 2k, \ldots, 4h+4k-2\} = \emptyset$. By the observation above, $4h+4k-2$ does not belong to one of S_x, S_y. First, assume that $4h+4k-2$ belongs to one of S_x, S_y, say, $4h+4k-2 \in S_x$. Then, $4h+4k-2 \notin S_y$. But $2k-1 \notin S_x$, since $(4h+4k-2)-(2k-1) = 2k-1+4h > |S_x|-1$, and then we can take $\{\alpha, \beta\} = \{2k-1, 4h+4k-2\}$. Finally, if $4h+4k-2 \notin S_x \cup S_y$, then we claim that either $2k-1$ or $3k-2$ does not belong to either S_x or S_y. In fact, if both numbers belong to both sets, then the interval $\{2k-1, 3k-2\}$ of size k would be contained in $S_x \cap S_y$. As $|S_x|, |S_y| \leq 2k-1$, this implies that $c(x), c(y) \in \{2k-1, 3k-2\}$, which contradicts $|c(x)-c(y)| \geq k$. This implies that we can take either $\{\alpha, \beta\} = \{2k-1, 4h+4k-2\}$ or $\{\alpha, \beta\} = \{3k-2, 4h+4k-2\}$. □

This theorem extends a result of [Colucci and Győri 2021] for the case $(h,k) = (2,1)$. In that paper, there is a construction that shows that the upper bound of Theorem 1 is sharp up to a multiplicative constant as well.

Theorem 2. *For every integers $h \geq 0$ and $k \geq 1$, there are oriented graphs G whose underlying graphs are biconnected, all in- and outdegrees bounded by d, and $\overrightarrow{\lambda}_{h,k}(G) \geq kd^2 + c_{h,k} \cdot d$ for some $c_{h,k}$ which depends only on h and k.*

3. Concluding remarks

In this paper, we proved that, for every pair of integers $h \geq 0$, $k \geq 1$, the $L(h,k)$-number of oriented graphs with bounded degree in every block of their underlying graph is bounded, which is a remarkable contrast with the $L(h,k)$-number of undirected graphs. On the other hand, in many cases, the upper bound given by Theorem 1 is not the exact value of the $L(h,k)$-number of the corresponding oriented graph. It would be interesting to determine the precise value, or at least better bounds, of this parameter in some classes of oriented graphs, as in [Chang et al. 2007] and [Chen et al. 2009].

References

Chang, G. J., Chen, J.-J., Kuo, D., and Liaw, S.-C. (2007). Distance-two labelings of digraphs. *Discrete applied mathematics*, 155(8):1007–1013.

Chang, G. J. and Liaw, S.-C. (2003). The $L(2,1)$-labeling problem on ditrees. *Ars Combinatoria*, 66:23–31.

Chen, Y.-T., Chia, M.-L., and Kuo, D. (2009). $L(p,q)$-labeling of digraphs. *Discrete applied mathematics*, 157(8):1750–1759.

Colucci, L. and Győri, E. (2021+). On $L(2,1)$-labelings of oriented graphs. *Discussiones Mathematicae Graph Theory*. In press, available at `https://www.dmgt.uz.zgora.pl/publish/bbl_view_press.php?ID=13093`.

Hale, W. K. (1980). Frequency assignment: Theory and applications. *Proceedings of the IEEE*, 68(12):1497–1514.

Yeh, K.-C. (1990). *Labeling graphs with a condition at distance two*. PhD thesis, University of South Carolina.