Some Partial Results on Linial's Conjecture for Matching-Spine Digraphs*

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Abstract. *Let k be a positive integer. A* partial k-coloring *of a digraph D is a set $\mathcal{C}$ of k disjoint stable sets and has* weight *defined as $\sum_{C\in\mathcal{C}}|C|$. An* optimal *$k$-coloring is a k-coloring of maximum weight. A* path partition *of a digraph D is a set $\mathcal{P}$ of disjoint paths of D that covers its vertex set and has* k-norm *defined as $\sum_{P\in\mathcal{P}}\min\{|P|,k\}$. A path partition $\mathcal{P}$ is* k-optimal *if it has minimum k-norm. A digraph D is* matching-spine *if its vertex set can be partitioned into sets X and Y, such that $D[X]$ has a Hamilton path and the arc set of $D[Y]$ is a matching. Linial (1981) conjectured that the k-norm of a k-optimal path partition of a digraph is at most the weight of an optimal partial k-coloring. We present some partial results on this conjecture for matching-spine digraphs.*

1. Introduction

For a digraph D, let $V(D)$ denote its vertex set and let $A(D)$ denote its arc set. Given an arc $a=(u,v)\in A(D)$, we say that u and v are *adjacent* and also that u is the *tail* and v is the *head* of a. The set of *neighbors* of a vertex u in D, denoted by $N(u)$, is the set of vertices that are adjacent to u. In this paper, we consider only digraphs without loops and parallel arcs. A *path* is a sequence of distinct vertices $P=(v_1,v_2,\ldots,v_\ell)$ such that $(v_i,v_{i+1})\in A(D)$ for every $1\leq i\leq \ell-1$. We denote by $V(P)$ the set of vertices of P and by $\mathrm{ter}(P)$ the vertex v_ℓ. We define the *order* of a path P, denoted by $|P|$, as its number of vertices. We denote by Pv_i the subpath $(v_1,v_2,\ldots,v_i)$ of P. A *Hamilton path* in D is a path containing every vertex in $V(D)$. A *cycle* is a sequence of vertices $C=(v_0,v_1,\ldots,v_\ell)$ with $\ell\geq 2$ such that $(v_i,v_{i+1})\in A(D)$ for every $0\leq i\leq \ell-1$, and all vertices are distinct except v_0 and v_ℓ which coincide. We say that a digraph D is *acyclic* if it does not contain a cycle. A digraph D is *transitive* if whenever $(u,v)\in A(D)$ and $(v,w)\in A(D)$, then $(u,w)\in A(D)$ as well.

A *path partition* $\mathcal{P}$ of a digraph D is a set of disjoint paths that cover $V(D)$. A path partition $\mathcal{P}$ of D is *optimal* if it has minimum cardinality, and we denote its cardinality by $\pi(D)$. Given a positive integer k, the *k-norm* of a path partition $\mathcal{P}$ of D, denoted by $|\mathcal{P}|_k$, is defined as $\sum_{P\in\mathcal{P}}\min\{|P|,k\}$. A path partition of D with minimum k-norm is called *k-optimal*, and we denote its k-norm by $\pi_k(D)$. Note that $\pi(D)=\pi_1(D)$.

A *stable set* S of a digraph D is a subset of pairwise non-adjacent vertices. We denote by $\alpha(D)$ the cardinality of a maximum stable set of D. Given a positive integer k, a *partial k-coloring* $\mathcal{C}$ of D is a set of k (possibly empty) disjoint stable sets. The *weight*

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of a partial k-coloring $\mathcal{C}$, denoted by $||\mathcal{C}||$, is defined as $\sum_{C\in\mathcal{C}} |C|$. A partial k-coloring $\mathcal{C}$ is *optimal* if it has maximum weight, and we denote its weight by $\alpha_k(D)$. Note that $\alpha(D) = \alpha_1(D)$.

[Dilworth 1950] proved that the equality $\pi(D) = \alpha(D)$ holds when D is a transitive acyclic digraph. [Gallai and Milgram 1960] generalized Dilworth's Theorem to arbitrary digraphs by relaxing the equality and establishing that $\pi(D) \leq \alpha(D)$ for every digraph D. [Greene and Kleitman 1976] generalized Dilworth's Theorem in a different way by showing that $\pi_k(D) = \alpha_k(D)$ for every transitive acyclic digraph D and every positive integer k. [Linial 1981] conjectured that Greene-Kleitman's Theorem can be generalized to arbitrary digraphs by relaxing the equality in the same way that Gallai-Milgram's result extends Dilworth's Theorem.

Conjecture 1 (Linial's Conjecture)**.** *Inequality $\pi_k(D) \leq \alpha_k(D)$ holds for every digraph D and every positive integer k.*

Some particular cases of Conjecture 1 are solved such as $k = 1$ (Gallai-Milgram's Theorem itself) and $k = 2$ [Berger and Hartman 2008]. It also holds for some particular classes of digraphs ([Linial 1981, Berge 1982]). The following two recent results on Linial's Conjecture are relevant to this paper. A digraph D is *arc-spine* if there is a partition $\{X, Y\}$ of $V(D)$ such that $D[X]$ has a Hamilton path and $D[Y]$ contains at most one arc; in particular, when $D[Y]$ is a stable set, we say D is *spine*. [Sambinelli et al. 2017] proved Linial's Conjecture for spine digraphs and [Yoshimura et al. 2019] extended such result for arc-spine digraphs. This paper is a direct follow-up of the latter.

A digraph D is *matching-spine* if there is a partition $\{X, Y\}$ of $V(D)$ such that $D[X]$ has a Hamilton path and the arc set of $D[Y]$ is a *matching*; clearly a superclass of arc-spine digraphs. We use the notation $D[X, Y]$ to indicate a partition with this property. In this paper we present partial results on Conjecture 1 for matching-spine digraphs.

2. Main results

Let $D[X, Y]$ be a matching-spine digraph, let P be a Hamilton path of $D[X]$ and let $k \geq 2$ be an integer. The *canonical path partition* $\mathcal{P}$ of D is the path partition consisting of P together with all maximal paths of $D[Y]$; clearly $\pi_k(D) \leq |\mathcal{P}|_k = \min\{|X|, k\} + |Y|$. Let Y^+ and Y^- be the subsets of vertices in Y which are, respectively, tail and head of some arc in $D[Y]$ and let $Y^0 = Y \setminus (Y^+ \cup Y^-)$. A *canonical partial k-coloring* $\mathcal{C}$ is a partial k-coloring consisting of stable sets $Y^0 \cup Y^-$, Y^+ and $\min\{k-2, |X|\}$ singletons of X; clearly $\alpha_k(D) \geq ||\mathcal{C}|| = |Y| + \min\{k-2, |X|\}$. Note that there is a gap of two to reach our desired inequality.

A matching-spine digraph $D[X, Y]$ is *k-tight* if $|X| \geq k$ and for every $S \subseteq X$, with $|S| = k$, one of the following holds: (1) there is a vertex $y \in Y$ such that $S \subseteq N(y)$, or (2) there is an arc $(y^+, y^-) \in A(D[Y])$ and there is a vertex $s \in S$ such that $S - \{s\} \subseteq N(y^+) \cap N(y^-)$ and $s \notin N(y^+) \cup N(y^-)$. An alternative way to state condition (2) is the following: s is the unique non-neighbor of y^+ and y^- in S. When $D[X, Y]$ is not k-tight it is *k-loose*. The motivation for splitting the class of matching spine digraphs into k-loose and k-tight is the fact that it is easy to obtain a partial k-coloring with weight $\min\{|X|, k\} + |Y|$ for the k-loose ones. This implies the following.

Proposition 2. *Let $k \geq 2$ be an integer and let $D[X, Y]$ be a k-loose matching-spine digraph. Then $\pi_k(D) \leq \alpha_k(D)$.*

Henceforth we assume that D is k-tight. Let $P = (x_1, \ldots, x_\ell)$ be a Hamilton path of $D[X]$. We say that P has a *zigzag* if one of the structures illustrated in Figure 1 exists. In Figure 1, y denotes some vertex in Y, (y^+, y^-) some arc in $D[Y]$ and i is some index such that $1 \leq i < \ell$. If P has a zigzag there is a path P' of order $|X| + 1$ and it is easy to show that $\pi_k(D) \leq |Y| + k - 1$. If P has no zigzag, then we say that P is *zigzag-free*.

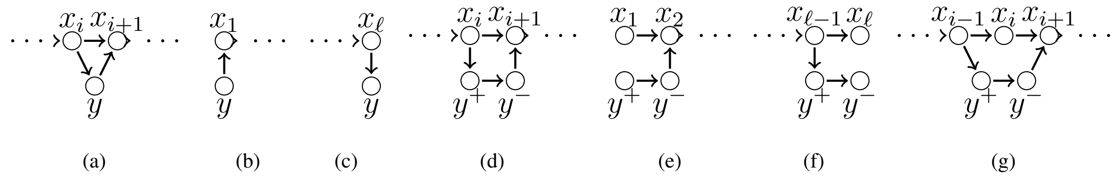

Figure 1. Types of zigzags of a path P.

Our main contribution in this paper is to show that $\pi_k(D) \leq |Y| + k - 1$ for every k-tight matching-spine digraph D (Theorem 4). To do so, we need Lemma 3. We present just a sketch of the proof for the sake of conciseness.

Lemma 3. *Let k be a positive integer, let $D[X, Y]$ be a k-tight matching-spine digraph and let $P = (x_1, x_2, \ldots, x_\ell)$ be a zigzag-free Hamilton path of $D[X]$. Then, there are paths P_1 and P_2 such that:* (i) $V(P_1) \cap V(P_2) = \emptyset$, (ii) $|P_1| + |P_2| = |X| + k + 1$, *and* (iii) $\mathrm{ter}(P_1) \cup \mathrm{ter}(P_2) = \{x_\ell, y\}$, *for some vertex $y \in Y$.*

Sketch of the proof. We may assume that Y is not a stable set, otherwise the result follows by [Sambinelli et al. 2017, Lemma 3]. The proof is by induction on k. The base case is $k \in \{1, 2\}$. When $k = 1$ take $P_1 = P$ and P_2 as some arc in $D[Y]$ and the result follows. Surprisingly, the case $k = 2$ is the longest part of the proof, requiring the analysis of many cases and is omitted. In the inductive step, we choose a vertex $x_i \in X$ such that $(x_i, y_i) \in A(D)$, for some $y_i \in Y$, and i is maximum with this property. Since P is zigzag-free, x_{i+1} exists and $x_{i+1} \notin N(y_i)$. Let $Y' = N(x_{i+1}) \cap Y$. By the choice of i, it follows that $(y', x_{i+1}) \in A(D)$, for every vertex $y' \in Y'$. Let $P' = Px_i$ and let $X' = V(P')$. We consider two cases. In case 1, we assume that there is no $y \in Y^+$ such that $(x_i, y) \in A(D)$. Let Y'' be the subset of vertices in Y^+ which have no neighbor in Y'. Let $D' = D[X' \cup Y' \cup Y'']$. We can prove that D' is a $(k-1)$-tight matching-spine digraph and P' is zigzag-free. In case 2, we assume that there is some $y \in Y^+$ such that $(x_i, y) \in A(D)$. In this case we define Y'' as follows. A vertex $y^+ \in Y^+$ belongs to Y'' if there is an arc $(y^+, y^-) \in A(D[Y])$ such that $y^+, y^- \in N(x_{i+2})$ and $y^- \notin Y'$. Let $D' = D[X' \cup Y' \cup Y'']$. We can prove that D' is a $(k-2)$-tight matching-spine digraph and P' is zigzag-free. In either case, we may apply the induction hypothesis to D' and P' to obtain P'_1 and P'_2 satisfying conditions (i)-(iii). We can extend P'_1 and P'_2 to paths P_1 and P_2 of D satisfying conditions (i)-(iii) as desired. □

Theorem 4. *Let $k \geq 2$ be an integer and let $D[X, Y]$ be a k-tight matching-spine digraph. Then, $\pi_k(D) \leq |Y| + k - 1$.*

Proof. Let P be a Hamilton path of $D[X]$. If P has a zigzag, then $\pi_k(D) \leq |Y| + k - 1$ and the result follows. So, we can assume that P is zigzag-free. By Lemma 3, there are paths P_1 and P_2 such that $|P_1| + |P_2| = |X| + k + 1$. Let $\mathcal{P}$ be a path partition consisting of P_1, P_2 and all trivial paths in $V(D) - (V(P_1) \cup V(P_2))$. Then, $|\mathcal{P}|_k = \min\{|P_1|, k\} + \min\{|P_2|, k\} + |V(D)| - |P_1| - |P_2| \leq k + k + |X| + |Y| - (|X| + k + 1) = |Y| + k - 1$. □

By Theorem 4, $\pi_k(D) \leq |Y| + k - 1$ for every k-tight matching-spine digraph D. Since $\alpha_k(D) \geq |Y| + k - 2$, we now have a gap of one to reach the inequality of Linial's Conjecture. In an attempt to close this gap, we introduce a subclass of k-tight matching-spine digraphs. A matching-spine digraph D is *k-strongly-tight* if X is a clique of size at least k and for every $T \subseteq X$ of size $k-1$ there is an arc $(y^+, y^-) \in A(D[Y])$ such that $T \subseteq (N(y^+) \cap N(y^-))$. We can show that if D is a matching-spine digraph which is not k-strongly-tight, then $\alpha_k(D) \geq |Y| + \min\{k-1, |X|\}$, which leads to the following theorem.

Theorem 5. *Let $k \geq 2$ be an integer and let $D[X, Y]$ be a matching-spine digraph which is not k-strongly-tight. Then, $\pi_k(D) \leq \alpha_k(D)$.*

3. Conclusion

We have proved Linial's Conjecture for matching-spine digraphs which are not k-strongly-tight (Theorems 4 and 5). This result relies on Lemma 3, whose proof is non-trivial and is the main contribution of this paper. Only the proof for k-strongly-tight matching-spine digraphs is missing in order to settle Linial's Conjecture for matching spine digraphs. We believe that inequality $\pi_k(D) \leq |Y| + k - 2$ holds for every k-strongly-tight matching-spine digraph. We have already proved the result for some cases that support this claim: for $k = 2$ and for $|X| = k + 1$.

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