The $(2,1)$-total number of near-ladder graphs*

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Abstract. *A $(2,1)$-total labelling of a simple graph G is a function $\pi\colon V(G) \cup E(G) \to \{0,\dots,k\}$ such that: $\pi(u) \neq \pi(v)$ for $uv \in E(G)$; $\pi(uv) \neq \pi(vw)$ for $uv, vw \in E(G)$; and $|\pi(uv) - \pi(u)| \geq 2$ and $|\pi(uv) - \pi(v)| \geq 2$ for $uv \in E(G)$. The $(2,1)$-total number $\lambda_2^t(G)$ of G is the least k for which G admits such a labelling. In 2008, Havet and Yu conjectured that $\lambda_2^t(G) \leq 5$ for every connected graph $G \ncong K_4$ with $\Delta(G) \leq 3$. We prove that, for near-ladder graphs, $\lambda_2^t(G) = 5$, verifying Havet and Yu's Conjecture for this class.*

1. Introduction

Let G be a simple graph with $V(G)$ its set of *vertices* and $E(G)$ its set of *edges*. The *degree* of $v \in V(G)$ is denoted $d(v)$ and the *maximum degree* of G is denoted $\Delta(G)$. If $d(v) = k$ for every $v \in V(G)$, then G is called *k-regular*; if $k = 3$, we say G is *cubic*.

The Frequency Channel Assignment Problem consists in the assignment of frequency channels to transmitters satisfying some given constraints. In fact, a network of transmitters can be viewed as a graph and the channels as colors or labels, which gives a graph theoretical perspective for the problem (Metzger, 1970; Zoeliner and Beall, 1977; Hale, 1980). A graph labelling that has been motivated by the Frequency Channel Assignment Problem is the $L(2,1)$-labelling, introduced by Griggs and Yeh (1992). An $L(2,1)$-*labelling* of a graph G is a function $f\colon V(G) \to \{0,\dots,k\}$ such that $|f(u) - f(v)| \geq 2$ for $uv \in E(G)$, and $f(u) \neq f(w)$ if $uv, vw \in E(G)$ and $uw \notin E(G)$. The least integer k for which a graph G admits an $L(2,1)$-labelling is denoted $\lambda_{2,1}(G)$. Fiala et al. (2001) proved that deciding whether $\lambda_{2,1}(G) \leq l$ is an NP-complete problem for all $l \geq 4$. However, $\lambda_{2,1}(G)$ has been determined for some classic families of graphs such as paths, cycles (Griggs and Yeh, 1992) and trees (Chang and Kuo, 1996).

In this work, we investigate a variant of $L(2,1)$-labellings, proposed by Havet and Yu (2008), called $(2,1)$-total labelling. A k-$(2,1)$-*total labelling* of a simple graph G is a function $\pi\colon V(G) \cup E(G) \to \{0,\dots,k\}$ such that: $\pi(u) \neq \pi(v)$ for $uv \in E(G)$; $\pi(uv) \neq \pi(vw)$ for $uv, vw \in E(G)$; and $|\pi(uv) - \pi(u)| \geq 2$ and $|\pi(uv) - \pi(v)| \geq 2$ for $uv \in E(G)$. The least integer k for which G admits a k-$(2,1)$-total labelling is denoted $\lambda_2^t(G)$ and called $(2,1)$-*total number*. We remark the strong relation between $L(2,1)$-labellings and $(2,1)$-total labellings: a (2,1)-total labelling of a graph G is associated with an $L(2,1)$-labelling of the graph obtained by subdividing each edge of G exactly once. The $(2,1)$-total number has been determined for cycles, paths, caterpillars (Havet and Yu, 2008; Khan et al., 2010), complete bipartite graphs (Lih et al., 2009), among

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others. When Havet and Yu (2008) introduced $(2,1)$-total labellings, they also proved that graphs with $\Delta(G) \leq 3$ have $\lambda_2^t(G) \leq 6$ and showed that this bound is tight, by proving that $\lambda_2^t(K_4) = 6$. However, the authors believe that K_4 is the only graph with $\Delta(G) = 3$ and $\lambda_2^t(G) > 5$ and, thus, posed Conjecture 1.

Conjecture 1. *If G is a connected graph with $\Delta(G) \leq 3$ and $G \ncong K_4$, then $\lambda_2^t(G) \leq 5$.*

Conjecture 1 has been verified for graphs G with $\Delta(G) \leq 2$ (Havet and Yu, 2008; Khan et al., 2010), but it is still open for graphs G with $\Delta(G) = 3$. Nevertheless, it was verified for a few classes of graphs with $\Delta(G) = 3$: for trees, Wang and Chen (2009) presented a necessary and sufficient condition for $\lambda_2^t(G) = 4$ or $\lambda_2^t(G) = 5$; for bipartite graphs, Havet and Thomassé (2009) presented a polynomial time algorithm which determines $\lambda_2^t(G)$; for Flower Snarks, Tong et al. (2010) showed that $\lambda_2^t(G) = 5$; for graphs whose vertex set is covered by a set of independent triangles, Sethuraman and Velankanni (2015) proved that $\lambda_2^t(G) = 5$.

In the article posing Conjecture 1, Havet and Yu (2008) established some initial properties and results. Two of these, important for this work, are presented in the following lemmas.

Lemma 2. *Let G be a simple $\Delta(G)$-regular graph. Then, $\lambda_2^t(G) \geq \Delta(G) + 2$.* □

Lemma 3. *Let G be a regular bipartite graph. Then $\lambda_2^t(G) = \Delta(G) + 2$.* □

In this work, we determine the $(2,1)$-total number of a family of cubic graphs called near-ladders, showing that these graphs verify Conjecture 1.

2. Main results

Initially, we define near-ladder graphs and present an auxiliary result. In Theorem 5, we show that near-ladder graphs have $(2,1)$-total number equal to five.

For an integer $\ell \geq 2$, let $n = 2\ell$. A *ladder* L_n is a simple graph with $V(L_n) = \{u_1, u_2, \ldots, u_\ell, v_1, v_2, \ldots, v_\ell\}$ and $E(L_n) = E_h \cup E_v$, such that $E_h = \{u_i u_{i+1}, v_i v_{i+1} : 1 \leq i < \ell\}$ and $E_v = \{u_i v_i : 1 \leq i \leq \ell\}$. A *near-ladder* is a cubic graph built from L_n, $n > 4$, by adding *laminar edges*, $v_1 v_\ell$ and $u_1 u_\ell$, or *cross edges*, $u_1 v_\ell$ and $v_1 u_\ell$. We partition the family of near-ladder graphs into two subfamilies PL_n and ML_n, according to the added edges. A graph PL_n (*Prism* or *Circular Ladder*) is obtained by adding laminar edges to L_n and, thus $V(PL_n) = V(L_n)$ and $E(PL_n) = E(L_n) \cup \{v_1 v_\ell, u_1 u_\ell\}$. A graph ML_n (*Möbius Ladder*) is obtained by adding cross edges to L_n. Thus, $V(ML_n) = V(L_n)$ and $E(ML_n) = E(L_n) \cup \{v_1 u_\ell, u_1 v_\ell\}$.

Lemma 4 characterizes the near-ladder graphs that are bipartite. Since these are cubic graphs, by Lemma 3, we conclude they verify Conjecture 1.

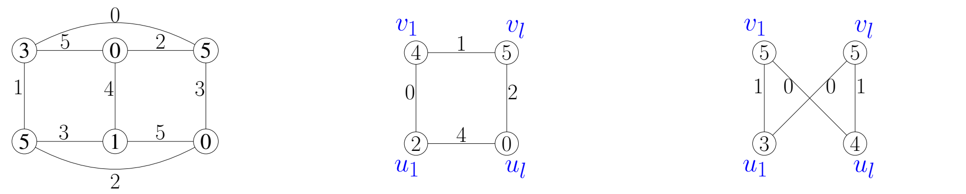
Lemma 4. *Let G be a near-ladder with $n = 2\ell$ and $\ell \geq 3$. If $G \cong PL_n$, then G is bipartite if and only if ℓ is even. If $G \cong ML_n$, then G is bipartite if and only if ℓ is odd.*

Outline of the proof. Let $G \cong PL_n$ with $n = 2\ell$, $\ell \geq 3$. Suppose G is bipartite. Then, since $v_1, v_2, \ldots, v_\ell$ induces a cycle, we conclude that ℓ is even. For $G \cong ML_n$, we consider cycle $v_1 \ldots v_\ell u_1 v_1$, which is odd when ℓ is even. For the converse, we show that $A = \{v_i : i \text{ is even}\} \cup \{u_i : i \text{ is odd}\}$ and $B = V(G) \setminus A$ is a bipartition of G for $G \cong PL_n$ when ℓ is even and for $G \cong ML_n$ when ℓ is odd. □

Theorem 5. *Let G be a near-ladder graph. Then, $\lambda_2^t(G) = 5$.*

Outline of the proof. Let G be a near-ladder graph with $n = 2\ell$, $\ell \geq 3$. If G is bipartite, then the result follows. Then, suppose G is nonbipartite. Thus, $G \cong PL_n$ with $n \equiv 2 \pmod 4$ or $G \cong ML_n$ with $n \equiv 0 \pmod 4$. In order to prove that $\lambda_2^t(G) = 5$ for these graphs, it is sufficient to present a 5-$(2,1)$-total labelling for them.

First, we describe, in Figure 1(a), a 5-$(2,1)$-total labelling for PL_6. Thus, for the remaining of this proof, suppose $G \ncong PL_6$. We consider two edge disjoint subgraphs of G: $B_{n-2} \cong L_{n-2}$, with $V(L_{n-2}) = \{v_2, v_3, \ldots, v_{\ell-1}, u_2, u_3, \ldots, u_{\ell-1}\}$; and $B_2 \cong L_2$ with $V(L_2) = \{v_1, v_\ell, u_1, u_\ell\}$. Subgraph B_2 has laminar or cross edges depending on whether $G \cong PL_n$ or $G \cong ML_n$. By construction, G is obtained from the union of B_{n-2} and B_2 by adding the *link edges* $E_L = \{u_1u_2, v_1v_2, u_{\ell-1}u_\ell, v_{\ell-1}v_\ell\}$. A 5-$(2,1)$-total-labelling π of G is obtained from 5-$(2,1)$-total labellings of B_{n-2} and B_2, and a specific label assignment for the edges of E_L.



(a) Graph PL_6. **(b) Subgraph B_2 of $G \cong PL_n$.** **(c) Subgraph B_2 of $G \cong ML_n$.**

Figure 1. 5-$(2,1)$**-total labellings of** PL_6 **and subgraphs** B_2**.**

Let $G \cong PL_n$, $n > 6$ and $n \equiv 2 \pmod 4$. For B_{n-2}, let π be defined as follows. For each vertex v_i, let $\pi(v_i) = 0$ if i is even, and $\pi(v_i) = 1$, otherwise. For each vertex u_i, let $\pi(u_i) = 1$ if i is even, and $\pi(u_i) = 0$, otherwise. For each edge u_iu_{i+1} and v_iv_{i+1}, let $\pi(u_iu_{i+1}) = \pi(v_iv_{i+1}) = 3$ if i is even, and $\pi(u_iu_{i+1}) = \pi(v_iv_{i+1}) = 4$, otherwise. For each edge u_iv_i, $3 \leq i \leq \ell - 1$, let $\pi(u_iv_i) = 5$, and $\pi(u_iv_i) = 4$ if $i = 2$. For B_2, consider the fixed 5-$(2,1)$-total labelling in Figure 1(b). Finally, the edges of E_L are assigned the following labels: $\pi(v_{\ell-1}v_\ell) = \pi(u_{\ell-1}u_\ell) = 3$, $\pi(v_1v_2) = 2$ and $\pi(u_1u_2) = 5$.

By the definition, every pair of adjacent vertices in B_{n-2} and every pair of adjacent edges in $E_h \cap E(B_{n-2})$ have distinct labels. Note that every $e \in E_h \cap E(B_{n-2})$ has $\pi(e) \in \{3, 4\}$, every $e \in E_v \cap E(B_{n-2}) \setminus \{u_2v_2\}$ has $\pi(e) = 5$, and $\pi(u_2v_2) = 4 \neq 3 = \pi(u_2u_3) = \pi(v_2v_3)$. Also, in B_{n-2}, the absolute difference between the label of an edge and the label of its endpoints is at least two since every vertex v has $\pi(v) \in \{0, 1\}$ and every edge e has $\pi(e) \in \{3, 4, 5\}$. This implies that π, restricted to B_{n-2}, is a 5-$(2,1)$-total labelling. Thus, recalling that B_2 was assigned a fixed labelling, we conclude that π, restricted to $G - E_L$, is a 5-$(2,1)$-total labelling of $G - E_L$. So, it remains to verify the labels assigned to the link edges.

By inspection, we first note that the endpoints of the link edges are assigned different labels. Now, we verify that the difference between the label of each link edge and the labels assigned to its endpoints is at least two. Since ℓ is odd: for edge $v_{\ell-1}v_\ell$, it holds that $|\pi(v_{\ell-1}v_\ell) - \pi(v_{\ell-1})| = |3-0| \geq 2$ and $|\pi(v_{\ell-1}v_\ell) - \pi(v_\ell)| = |3-5| \geq 2$; for edge $u_{\ell-1}u_\ell$, we have $|\pi(u_{\ell-1}u_\ell) - \pi(u_{\ell-1})| = |3-1| \geq 2$ and $|\pi(u_\ell v_{\ell-1}) - \pi(u_\ell)| = |3-0| \geq 2$. Also, $|\pi(v_1v_2) - \pi(v_1)| = |2-4| \geq 2$ and $|\pi(v_1v_2) - \pi(v_2)| = |2-0| \geq 2$ for edge v_1v_2. Finally, for edge u_1u_2, it holds that $|\pi(u_1u_2) - \pi(u_1)| = |5-2| \geq 2$ and $|\pi(u_1u_2) - \pi(u_2)| = |5-1| \geq 2$.

Now, we verify the labels of the edges adjacent to the link edges. For $x \in V(G)$, let $L(x)$ be the set of labels of the non-link edges incident with x. By construction: $L(v_2) = L(u_2) = \{3, 4\}$; since ℓ is odd, $L(v_{\ell-1}) = L(u_{\ell-1}) = \{4, 5\}$; $L(v_1) = \{0, 1\}$; $L(u_1) = \{0, 4\}$; $L(v_\ell) = \{1, 2\}$; and $L(u_\ell) = \{2, 4\}$. Thus, $\pi(v_{\ell-1}v_\ell) = 3 \notin L(v_{\ell-1}) \cup L(v_\ell)$, $\pi(u_{\ell-1}u_\ell) = 3 \notin L(u_{\ell-1}) \cup L(u_\ell)$, $\pi(v_1v_2) = 2 \notin L(v_1) \cup L(v_2)$ and $\pi(u_1u_2) = 5 \notin L(u_1) \cup L(u_2)$. Therefore, we conclude that π is a 5-$(2, 1)$-total labelling of G.

In order to complete the proof, consider $G \cong ML_n$ with $n \equiv 0 \pmod 4$. For this case, the vertices of B_{n-2} are labelled as in the previous case and its edges are labelled as follows. For each edge u_iu_{i+1} and v_iv_{i+1}, let $\pi(u_iu_{i+1}) = \pi(v_iv_{i+1}) = 4$ if i is even, and $\pi(u_iu_{i+1}) = \pi(v_iv_{i+1}) = 3$, otherwise. For each edge u_iv_i, $3 \leq i \leq l-1$, let $\pi(u_iv_i) = 5$, and $\pi(u_iv_i) = 3$ if $i = 2$. For B_2, consider the fixed 5-$(2, 1)$-total labelling presented in Figure 1(c). Finally, the labels of the edges of E_L are defined as follows: $\pi(v_{\ell-1}v_\ell) = 3$, $\pi(u_{\ell-1}u_\ell) = \pi(v_1v_2) = 2$ and $\pi(u_1u_2) = 5$. We verify that π is a 5-$(2, 1)$-total labelling by a reasoning similar to the one used for PL_n, $n > 6$ and $n \equiv 2 \pmod 4$. $\square$

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