# Edge Intersection Graphs of Paths on a Triangular Grid

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***Abstract.*** *We introduce a new class of intersection graphs, the edge intersection graphs of paths on a triangular grid, called EPG$_t$ graphs. We compare this new class with the well-known class of EPG graphs. A turn of a path at a grid point is called a* bend. *An EPG$_t$ representation in which every path has at most $k$ bends is called a $B_k$-EPG$_t$ representation and the corresponding graphs are called $B_k$-EPG$_t$ graphs. We characterize the representation of cliques with three vertices and chordless 4-cycles in $B_1$-EPG$_t$ representations.*

## 1. Introduction

In 2009, Golumbic, Lipshteyn and Stern [Golumbic et al. 2009] introduced the notion of edge intersection graphs of paths on a rectangular grid. This family of graphs, called EPG graphs, is a generalization of the edge intersection graphs of paths on a degree-four tree [Golumbic and Jamison 1985, Golumbic et al. 2005], in the sense that every EPT representation on a degree-four tree is an EPG representation. We consider here an even more general structure, from which a family of paths is taken, the triangular grid. A triangular grid consists of a rectangular grid with an extra direction (see Figure 1). We call this extra direction the *diagonal*. In most applications, a triangular grid is usually displayed as depicted in Figure 1(a). Here, however, it is more natural to consider it depicted as in Figure 1(b), since we are treating such a grid as a generalization of the rectangular one. Notice that both drawings are equivalent, in the sense that if we rotate 15 degrees the "/"-shaped lines, and 30 degrees the "\"-shaped ones, both in counter-clockwise direction, of Figure 1(a), we obtain precisely the grid drawing of Figure 1(b). We call the edge intersection graphs of paths on a triangular grid as EPG$_\mathrm{t}$ graphs.

A motivation for studying these graphs is the same from EPG graphs, coming originally from circuit layout problems [Molitor 1991]. Another motivation is a rather natural optimization one, which consists of deciding whether an EPG$_\mathrm{t}$ graph admits a representation having paths bending at most $k$ times. In this paper, we introduce this new class of EPG$_\mathrm{t}$ graphs and provide a characterization of representations of

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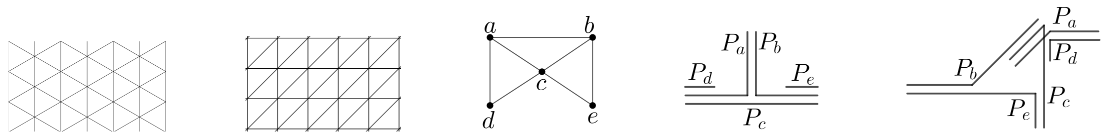
cliques of size 3 and 4-cycles on those grids, extending the analogous results for EPG graphs [Golumbic et al. 2009].

The triangular grid has been studied in the context of the *channel assignment problem with separation (CAPS)*. In cellular networks, a large number of base stations are expected to cover communications over a region [Zander 2000]. Such a covering can be achieved by placing base stations according to a regular plane tessellation. The most important regular tessellation of the plane is triangular tessellation [Bertossi et al. 2004], and the corresponding topology of such a tessellation is the triangular grid, known as *triangular lattice* on those applications. The reason for adopting this particular tessellation comes from the fact that base stations are uniformly distributed in the coverage region, and an individual base station generally has six directional transceivers [Janssen et al. 1998]. Thus, the base station's coverage area can be idealized as a regular triangular tessellation. The channel assignment problem with separation (CAPS) deals with assigning frequencies to stations such that there is no interference between frequencies assigned to nearby stations while trying to minimize the span (the difference between highest and lowest frequencies) of assigned frequencies.




(a) (b) (c) (d) (e)

**Figure 1. (a-b) The triangular grid, (c) graph, (d) $B_1$-EPG representation and (e) $B_1$-EPG$_t$ representation.**

Let $P$ and $Q$ be paths on a triangular grid $\mathcal{G}$. We write $P \cap_v Q$ when referring to the vertex intersection between $P$ and $Q$, and $P \cap_e Q$ when referring to the edge intersection between $P$ and $Q$. We define the *edge intersection graph of paths* EPG$_t(\mathcal{P})$ of a collection of paths $\mathcal{P}$ on a triangular grid $\mathcal{G}$ as having vertices which correspond to the members of $\mathcal{P}$, such that two vertices are adjacent in EPG$_t(\mathcal{P})$ if and only if the corresponding paths in $\mathcal{P}$ share at least one edge on $\mathcal{G}$. A graph $G$ is called an *edge intersection graph of paths on a triangular grid* (EPG$_t$) if $G =$ EPG$_t(\mathcal{P})$ for some $\mathcal{P}$ and $\mathcal{G}$, and $\langle \mathcal{P}, \mathcal{G} \rangle$ is an EPG$_t$ *representation* of $G$. Similarly to the EPG graphs, a turn of a path at a grid point is called a *bend*. A path is a *$B_k$-path* if it contains at most $k$ bends. An EPG$_t$ representation is *$B_k$-EPG$_t$* if each path has at most $k$ bends. A graph that has a $B_k$-EPG$_t$ representation is called *$B_k$-EPG$_t$*. The *triangular bend-number* of a graph $G$ is the least $k$ such that $G$ is $B_k$-EPG$_t$. The graph in Figure 1(c) is $B_1$-EPG$_t$, as the representation in Figure 1(e) shows.

A *segment* of a path is a maximal subpath of the path with no bends. Therefore, a 0-bend path has only one segment (the path itself), whereas a 1-bend path has two segments. A 1-bend path can be referred to as *narrow*, *normal* or *wide*, depending on the angle formed by its two segments. Note in Figure 1(e) that $P_a$ and $P_b$ are wide paths, $P_d$ and $P_e$ are normal paths and $P_c$ is a narrow path.

Let $G$ be a graph and $\langle \mathcal{P}, \mathcal{G} \rangle$ a $B_1$-EPG$_t$ representation of $G$ on a triangular grid $\mathcal{G}$, where $\mathcal{P} = \{P_i \mid 1 \leq i \leq |V(G)|\}$. We define $U(\mathcal{P}) \subset \mathcal{G}$ as the paths of $\mathcal{P}$, such that: $U(\mathcal{P}) = \{s \mid s$ is a segment of a path $P_i$ such that $P_i \cap_e P_j \neq \emptyset$

for some $1 \leq i, j \leq |V(G)|$ with $i \neq j\}$ that is, $U(\mathcal{P})$ is the subgraph of $\mathcal{G}$ induced by the vertices in segments of paths which intersect other paths in the family.

## 2. Cliques on $\mathbf{B_1}$-$\mathbf{EPG_t}$ Representations

In this section, we characterize the $B_1$-$EPG_t$ representations of cliques with three vertices.

Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a $B_1$-$EPG_t$ representation of a graph $G$ on a triangular grid $\mathcal{G}$. Let $C$ be a maximal clique of $G$ and $\mathcal{P}_C \subseteq \mathcal{P}$ be the set of paths representing the vertices of $C$. If $\bigcap_e \mathcal{P}_C \neq \emptyset$, then $C$ is called an *edge-clique*. If $\bigcap_e \mathcal{P}_C = \emptyset$ and $\bigcap_v \mathcal{P}_C = \{b\}$, then $C$ is called a *claw-clique*. If $U(\mathcal{P}_C)$ has a right triangle $T$ as a subgraph, then $C$ is called a *triangular-clique*.

Let $T$ be a right triangle on the grid and $s_1$, $s_2$ and $s_3$ the sides of $T$. The points of the grid $\{v_1\} = s_1 \cap_v s_2$, $\{v_2\} = s_1 \cap_v s_3$ and $\{v_3\} = s_2 \cap_v s_3$ are called the *corners* of $T$.

Note that the existence of a third direction on the grid allows the arising of a new type of clique, the triangular clique. See Figure 2 for examples of it.

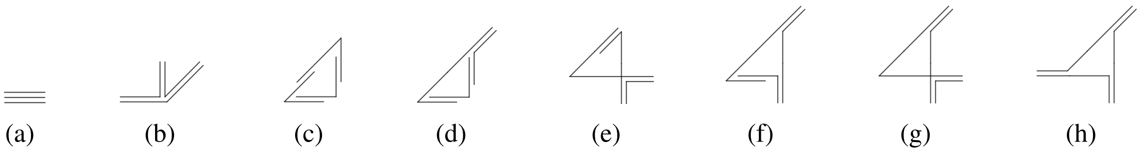


**Figure 2. (a) Edge-clique, (b) claw-clique and (c-h) triangular-clique.**

**Theorem 1.** *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a $B_1$-$EPG_t$ representation of a graph $G$, and let $C = \{v_1, v_2, v_3\}$ be a clique of $G$. Then, $C$ corresponds to either an edge-clique, a claw-clique, or a triangular-clique.*

## 3. Cycles on $\mathbf{B_1}$-$\mathbf{EPG_t}$ Representations

In this section, we characterize the $B_1$-$EPG_t$ representations of 4-cycles.

Let $G$ be a chordless 4-cycle and $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ the set of paths representing the vertices of $G$ on a triangular grid. We denote by $P_{x,y}$ a path of $G$ connecting vertices $x$ and $y$. If $U(\mathcal{P})$ is a subdivision of a 4-star, let $b$ be its central vertex and $a_1, a_2, a_3, a_4$ be the vertices of $U(\mathcal{P})$ that have degree 1. Consider the following cases:

- If each $P_{a_i,b} \cup P_{a_{i+1},b}$ for all $1 \leq i \leq 4$ is contained in a different member of $\mathcal{P}$, where $a_5 = a_1$, then $\mathcal{P}$ is called a *true pie*. In a true pie, at least three of the four paths bend at $b$. See Figure 3(a).
- If each $P_{a_1,b} \cup P_{a_2,b}$, $P_{a_2,b} \cup P_{a_4,b}$, $P_{a_4,b} \cup P_{a_3,b}$, $P_{a_3,b} \cup P_{a_1,b}$ is contained in a different member of $\mathcal{P}$, then $\mathcal{P}$ is called a *false pie*. In a false pie, at least two of the paths bend at $b$. See Figure 3(b).

Let $Q$ be a quadrilateral subgraph of $\mathcal{G}$ of any size, and let $s_1$, $s_2$, $s_3$, $s_4$ be the segments of $\mathcal{G}$ forming the sides of $Q$, such that $s_i \cap_v s_{i+1} \neq \emptyset$ for $1 \leq i \leq 4$, where $s_5 = s_1$. We call $s_i \cap_v s_{i+1}$ the *corners* of $Q$. If $Q$ is a subgraph of $U(\mathcal{P})$, each corner of $Q$ is the bend for a different member of $\mathcal{P}$, $P_2 \cap_e P_3 \neq \emptyset$, $P_3 \cap_e P_4 \neq \emptyset$, $P_4 \cap_e P_1 \neq \emptyset$, $P_2 \cap_e P_4 = \emptyset$, and $P_1 \cap_e P_3 = \emptyset$, then $\mathcal{P}$ is called a *frame*.

Let $T$ be a right triangle in $\mathcal{G}$. If $T \subseteq U(\mathcal{P})$, each corner of $T$ is the bend for at most two different members of $\mathcal{P}$, $P_2 \cap_e P_3 \neq \emptyset$, $P_3 \cap_e P_4 \neq \emptyset$, $P_4 \cap_e P_1 \neq \emptyset$, $P_2 \cap_e P_4 = \emptyset$, and $P_1 \cap_e P_3 = \emptyset$, then $\mathcal{P}$ is called a *flag*. See Figure 3(f).

Let $T_1$, $T_2$ be distinct right triangles, such that $T_1 \cap_v T_2 = \{v\}$ where $v$ is a corner of both $T_1$ and $T_2$. If $G = T_1 \cup T_2 \subseteq U(\mathcal{P})$, each corner of $G$ is the bend of a different member of $\mathcal{P}$, $P_2 \cap_e P_3 \neq \emptyset$, $P_3 \cap_e P_4 \neq \emptyset$, $P_4 \cap_e P_1 \neq \emptyset$, $P_2 \cap_e P_4 = \emptyset$, and $P_1 \cap_e P_3 = \emptyset$, then $\mathcal{P}$ is called a *butterfly*. See Figure 3(g).

Note that the existence of a third direction on the grid, when compared to a rectangular grid, allows the arising of new representations of a 4-cycle. See in Figure 3 some examples of representations of a 4-cycle on a triangular grid.

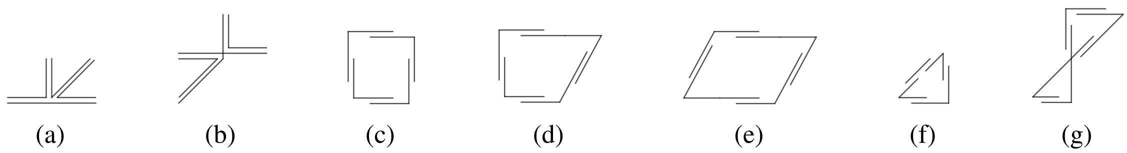



**Figure 3. (a) True pie, (b) false pie, (c-e) frame, (f) flag and (g) butterfly.**

**Theorem 2.** *Let $\langle \mathcal{P}, \mathcal{G} \rangle$ be a $B_1$-EPG$_t$ representation of G. Every chordless 4-cycle in G corresponds to either a true pie, a false pie, a frame, a flag or a butterfly in $\mathcal{P}$.*

## 4. Conclusions and Open Questions

We introduce the concept of $\mathrm{B}_k$-EPG$_\mathrm{t}$ graphs, a generalization of $\mathrm{B}_k$-EPG graphs. We characterize the representation of cliques of size 3 and chordless 4-cycles in $\mathrm{B}_1$-EPG$_\mathrm{t}$ graphs and, we conjecture that the representation of maximal cliques in $\mathrm{B}_1$-EPG$_\mathrm{t}$ graphs can be characterized by the edge-clique, claw-clique and triangular-clique. The complexity of recognizing $\mathrm{B}_k$-EPG (resp. $\mathrm{B}_k$-EPG$_\mathrm{t}$) graphs is open for all $k \geq 3$ (resp. $k \geq 1$).

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