Three Questions about Equitable Total Coloring of Small Cubic Graphs*

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Abstract. *A total coloring assigns colors to the vertices and edges of a graph without conflicts and it is called* equitable *if the cardinalities of any two color classes differ by at most 1. In 2020, Stemock considered equitable total colorings of small cubic graphs and conjectured that every* 4*-total coloring of a cubic graph with less than* 20 *vertices is equitable. We present counterexamples to Stemock's conjecture. We determine that every* 4*-total coloring must be equitable on all cubic graphs with* 6*,* 8*,* 10*, and* 14 *vertices. On the other hand, for cubic graphs with* 12*,* 16*, and* 18 *vertices, we characterize the color class configurations that might allow a non-equitable 4-total coloring. We prove that a cubic graph of* 12 *vertices is the smallest counterexample to Stemock's conjecture.*

1. Stemock's Conjecture

A total coloring of a graph G is a color assignment to the set of all elements $E(G) \cup V(G)$, where $E(G)$ denotes the set of edges and $V(G)$ denotes the set of vertices of the graph, in a set of colors $C = \{c_1, c_2, ..., c_k\}$, $k \in \mathbb{N}$, so that different colors are assigned to: every pair of vertices that are adjacent, all edges that are adjacent, and, each vertex and its incident edges. Behzad and Vizing [Behzad 1965, Vizing 1964] independently conjectured the Total Coloring Conjecture (TCC) that for any simple graph G, we have $\chi''(G) \leq \Delta(G) + 2$. If $\chi''(G) = \Delta(G) + 1$, then the graph is called Type 1; if $\chi''(G) = \Delta(G) + 2$, then the graph is called Type 2. A total coloring is called equitable if the cardinalities of any two color classes differ from at most 1.

A graph G is called regular of degree r, or r-regular when all its vertices have the same degree r. The 3-regular graphs are called cubic graphs. In 2020, Stemock established a conjecture that motivated our work [Stemock 2020].

Conjecture 1 ([Stemock 2020])**.** *Every* 4*-total coloring of a cubic graph* G *is equitable, given that the number of vertices of* G *is less than 20.*

This upper bound was motivated by the Type 1 graph R with 20 vertices found in [Dantas et al. 2016] that does not have an equitable total coloring with 4 colors (see Figure 1). Note that the color 4 appears 14 times and the others colors appear 12 times.

After some research, we concluded that Conjecture 1 is false. We reached this conclusion by consulting two articles. In [Chetwynd and Hilton 1988], the authors present a

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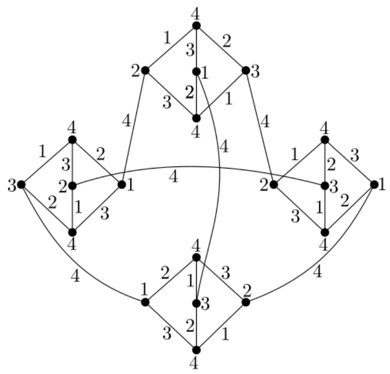

Figure 1. A non-equitable 4-total coloring of graph R.

method to obtain the total coloring of circular ladder graphs that generates a non-equitable 4-total coloring to the circular ladder graphs with 12 and 18 vertices. Subsequently, we were able to find equitable 4-total colorings for those graphs, see Figures 2 and 3.

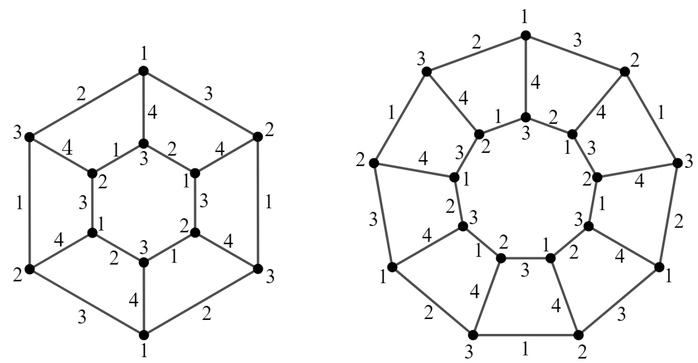

Figure 2. Circular ladder graphs L_{12} and L_{18} with non-equitable 4-total colorings.

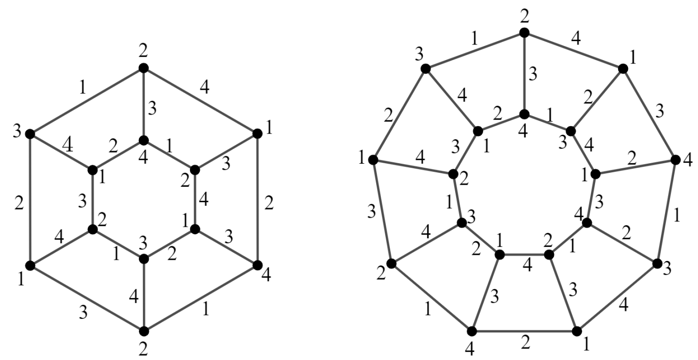

Figure 3. Circular ladder graphs L_{12} and L_{18} with equitable 4-total colorings.

We continued to study which cubic graphs with less than 20 verified Stemock's conjecture. Thus, we obtained the results that will be presented in the next two sections.

In addition, in all the results obtained, we used the conformable condition. Let G be a regular graph. G is conformable if and only if it has a vertex coloring with $\Delta(G) + 1$ colors and each color class has the same parity of $|V(G)|$. If G is a Type 1 graph, then G is conformable [Chetwynd and Hilton 1988].

2. Our all Equitable Results, when the graph has 6, 8, 10 or 14 vertices

We analyze the number of elements of a cubic graph, when the graph has less than 20 vertices, according to Stemock's conjecture. For each case, we study the largest total independent set of elements. A set of vertices A is called total independent set if no two vertices in set A are adjacent to each other, no two edges in set A share a vertex, and no vertex is the extreme of an edge in set A. Note that K_4 and the Möbius ladder M_6 are both Type 2 and both have an equitable 5-total coloring. The circular ladder L_6 has an equitable 4-total coloring [Dantas et al. 2016].

Theorem 1 ([Adauto 2022])**.** *All 4-total colorings of Type 1 cubic graphs with* 6*,* 8*,* 10*, and* 14 *vertices are equitable.*

Proof. For any number of vertices in this theorem, the proof strategy is the same. We divide the number of elements in the graph by 4 and check the configuration of the color classes of an equitable 4-total coloring. Next, we take the color class with the highest cardinality as a parameter and assume that there is no total independent set greater than this color class. If this is verified, we guarantee that the total coloring is equitable. To give an idea of the argument, we consider the case with 10 vertices.

Note that a cubic graph of 10 vertices has 15 edges. Thus, in every equitable 4-total coloring of these graphs each color class has exactly 7, 6, 6, and 6 elements respectively. Furthermore, the number of vertices in each color class will necessarily be even, since the graph is conformable. If there is a total independent set of 8 elements or two disjoint total independent sets of 7 elements it would be possible to assign a color to these sets and the coloring might not be equitable. However, we will prove that on a cubic graph of 10 vertices a total independent set of 8 elements does not exist and that two total independent sets of 7 elements cannot exist simultaneously. We divide the proof into cases according to the number of elements of each kind in a total independent set of 8 elements or two total independent sets of 7 elements. The possible cases are: 8 vertices and 0 edges, 6 vertices and 2 edges, 4 vertices and 4 edges, 2 vertices and 6 edges, 0 vertices and 8 edges, two disjoint total independent sets with 7 elements: 4 vertices and 3 edges each.

For instance, for the case with 6 vertices and 2 edges, there are 18 edges joining the set of 6 vertices to the remaining set of 4 vertices. But these 4 vertices have already one neighbor inside this set, therefore, this set can only receive at most 8 edges, a contradiction. Furthermore, for the case with two disjoint total independent sets with 7 elements, consider one total independent set with 7 elements consisting of vertices v_1, v_2, v_3, v_4 and the edges e_1, e_2, e_3 where the vertices a_1, a_2, a_3, a_4, a_5 and a_6 are the extremes of the edges e_1, e_2 and e_3 respectively (see Figure 4). For the other disjoint total independent set with 7 elements we must select two vertices among one of the extremes of each edge e_1, e_2, and e_3 together with 5 edges defining a matching, which is impossible. □

3. Our non-Equitable Results, when the graph has 12, 16 or 18 vertices

We first make a study of cubic graphs with 12 or 18 vertices, in light of the counterexamples L_{12} and L_{18} of the Stemock's conjecture. Please see in Figure 2. We were able to characterize the color class configurations that might allow a non-equitable 4-total coloring for all graphs with 12 or 18 vertices. Although the 4-total coloring given in [Chetwynd and Hilton 1988] for L_{16} is equitable, to consider all small graphs up to 20

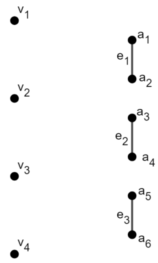

Figure 4. Diagram of case with 4 vertices and 3 edges in a total independent set of 7 elements, considering a graph with $n = 10$ vertices.

vertices, we were able to characterize the color class configurations that might allow a non-equitable 4-total coloring for all graphs with 16 vertices as well.

Theorem 2 ([Adauto 2022])**.** *The color classes configurations that might allow a non-equitable 4-total coloring of a cubic graph of $12, 16, 18$ vertices are respectively* 8, 8, 8, 6*;* 12, 12, 12, 9 *or* 12, 12, 10, 10*;* 11, 10, 10, 9.

4. Current Work

When we verified that the Stemock's conjecture was false, we asked three questions and only answered one of them. Our goal is to answer the remaining two.

Question 1. *Does every Type 1 cubic graph, with less than 20 vertices, have at least one equitable 4-total coloring?*

So far, for all cubic graphs with less than 20 vertices that we have obtained a non-equitable 4-total coloring, we have also obtained an equitable 4-total coloring.

Question 2. *Is the graph L_{12} the smallest counterexample to the Conjecture 1?*

Yes, it is the smallest. This is verified by the theorems obtained.

Question 3. *What is the largest value of n, such that every Type 1 cubic graph with n vertices is such that all of its 4-total colorings are equitable?*

For now, the largest n we have obtained for these conditions is 14. We seek in future works to answer whether in fact it is the greatest. Furthermore, for cubic graphs that can admit a non-equitable 4-total coloring, we want to determine the color class configurations of these colorings.

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