Seymour's Second Neighborhood Conjecture on sparse random graphs

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Abstract. *Seymour's Second Neighborhood Conjecture (SNC) says that every oriented graph contains a vertex whose second neighborhood is at least as large as its first neighborhood. We prove that asymptotically almost surely (a.a.s.) every orientation of the binomial random graph $G(n,p)$ satisfies the SNC if $n^4p^6 \to 0$. We also show that if $p \in (0, 2/3)$, then a.a.s. a uniformly-random orientation of $G(n,p)$ satisfies the SNC, settling it for almost every labeled oriented graph.*

Resumo. *A Conjectura da Segunda Vizinhança de Seymour (SNC) diz que todo grafo orientado contém um vértice cuja segunda vizinhança tem pelo menos tantos vértices quanto sua primeira vizinhança. Nós provamos que assintoricamente quase certamente (a.q.c.) toda orientação do grafo aleatório binomial $G(n,p)$ satisfaz à SNC quando $n^4p^6 \to 0$. Mostramos também que se $p \in (0, 2/3)$ é uma constante, então a.q.c. uma orientação uniformemente aleatória de $G(n,p)$ satisfaz à SNC, confirmando a conjectura para quase todo grafo orientado rotulado.*

An *oriented graph* D is a directed graph (or digraph) obtained from a simple graph G by assigning a direction to each of its edges (i.e., D contains no loops, parallel arcs, nor directed cycles of length 2); we also say that D is an *orientation* of G. Given a positive integer i, the *i-th neighborhood* of a vertex u, denoted by $N^i(u)$, is the set of vertices v for which a shortest directed path from u to v has precisely i arcs. A *Seymour vertex* (see [Seacrest 2015]) is a vertex u for which $|N^2(u)| \geq |N^1(u)|$. In 1990, Seymour conjectured that every oriented graph contains such a vertex (see [Dean and Latka 1995]).

Conjecture 1. *Every oriented graph contains a Seymour vertex.*

Conjecture 1, also known as Seymour's Second Neighborhood Conjecture (SNC), has been intensively studied (see, e.g., [Fidler and Yuster 2007, Seacrest 2015, Chen et al. 2003]). Notably, it was confirmed for tournaments (orientations of complete graphs) using two distinct approaches [Fisher 1996, Havet and Thomassé 2000]; and it was also studied in a random digraph model (in which each ordered pair is independently chosen to be an arc with probability $p < 1/2$) [Cohn et al. 2016].

Indeed, the study of random analogues of combinatorial problems has been a fruitful area of research (see [Conlon and Gowers 2016]). We study Conjecture 1 for arbitrary orientations of the (undirected) binomial random graph $G(n, p)$, defined as the random graph with vertex set $\{1, \ldots, n\}$ in which each pair of vertices appears independently as an edge with probability p. Note that Conjecture 1 holds when $G(n, p)$ is sufficiently sparse since $G(n, p)$ asymptotically almost surely (a.a.s.) has an isolated vertex whenever $np \leq (1 - \varepsilon) \log n$. Our first result (see Section 1), extends this to a larger range of p.

Theorem 2. *Let $p : \mathbb{N} \to (0, 1)$. If $n^4 p^6 \to 0$, then a.a.s. every orientation of $G(n, p)$ contains a Seymour vertex.*

We also confirm that Conjecture 1 holds a.a.s. when $0 < p < 2/3$ and D is an orientation of $G(n, p)$ whose minimum outdegree $\delta^+(D)$ is large, implying that almost every orientation of almost every n-vertex graph contains a Seymour vertex (see Section 2).

Lemma 3. *If $p \in (0, 2/3)$ is a fixed constant, then a.a.s. every orientation D of $G(n, p)$ which satisfies $\delta^+(D) \geq \sqrt{8n^{3/2}/(1 - \frac{3}{2}p)}$ has a Seymour vertex.*

1. Wheel-free graphs

A *wheel* is a graph obtained from a cycle C by adding a new vertex adjacent to all vertices in C. We prove Theorem 2 by showing that sufficiently sparse random graphs are wheel-free, and then arguing that orientations of wheel-free graphs contain a Seymour vertex.

Lemma 4. *If $p, \varepsilon \in (0, 1)$ and $n^4 p^6 < \varepsilon/16$, then $\mathbb{P}\big(G(n, p)$ is wheel-free$\big) \geq 1 - \varepsilon$.*

Proof. Since $n^4 p^6 < \varepsilon/16$, we have that

$$np^2 < (\varepsilon p^2/16)^{1/4} < 1/2. \tag{1}$$

Let X_k be the number of wheels of order k in $G(n, p)$, and let $X = \sum_{k=4}^{n} X_k$ be the number of wheels in $G(n, p)$. We have

$$\begin{aligned}\mathbb{E}X &= \sum_{k=4}^{n} \mathbb{E}X_k = \sum_{k=4}^{n} \binom{n}{k} k \frac{(k-1)!}{2(k-1)} p^{2(k-1)} \\ &< n \sum_{k=4}^{n} (np^2)^{k-1} = n^4 p^6 \sum_{k=0}^{n-4} (np^2)^k \overset{\text{G.P.}}{<} \frac{n^4 p^6}{1 - np^2} \overset{(1)}{<} \varepsilon.\end{aligned} \tag{2}$$

Where in (2) we used the standard estimate for the sum of a geometric progression (G.P.) with ratio $np^2 < 1$. By Markov's inequality, we have $\mathbb{P}(X \geq 1) \leq (\mathbb{E}X)/1 < \varepsilon$. □

To show that every orientation of a wheel-free graph has a Seymour vertex, we prove a slightly stronger result. A subset A of $V(D)$ is *cornering* if it induces a digraph with minimum outdegree 0; we say that a digraph D is *locally cornering* if $N^1(u)$ is cornering for every $u \in V(D)$. In particular, if D is locally cornering and u is a vertex of minimum outdegree in D, then a vertex v with outdegree 0 in $N^1(u)$ has at least $\delta^+(D) = |N^1(u)|$ outneighbors outside $N^1(u)$, and hence u is a Seymour vertex.

Lemma 5. *Every locally cornering digraph has a Seymour vertex.*

Lemmas 4 and 5 yield the following corollary, which implies Theorem 2.

Corollary 6. *For all positive $p, \varepsilon \in (0,1)$ and all $n \in \mathbb{N}$, if $n^4p^6 < \varepsilon/16$, then the probability that every orientation of $G(n,p)$ has a Seymour vertex is at least $1-\varepsilon$.*

Proof. Note that every orientation of a wheel-free graph is locally cornering, because in such a graph the neighborhood of every vertex is a forest, and every forest has a vertex with no outneighbors. The result is then immediate from Lemmas 4 and 5. □

2. Orientations with prescribed minimum degree

In this section we show that if p is a constant in $(0, 2/3)$, then a.a.s. a typical orientation of $G(n,p)$ has a Seymour vertex. More precisely, we use the following Chernoff-type bound to show that orientations of $G(n,p)$ with minimum degree $\Omega(n^{1/2})$ have a Seymour vertex.

Lemma 7 (Chernoff inequality). *Let X be a binomial random variable $\mathcal{B}(N,p)$, i.e., the number of successes in N mutually independent experiments with success $p \in (0,1)$ each. For all $t > 0$ we have $\mathbb{P}\big(|X - \mathbb{E}X| > t\big) < 2\exp\big(\frac{-t^2}{2(\sigma^2+t/3)}\big)$, where $\sigma^2 = Np(1-p)$ denotes the variance of X.*

For $X, Y \subseteq V(G)$, let $e(X,Y) = \big|\{e \in E(G) : |X \cap e| = |Y \cap e| = 1\}\big|$ and $e(X) = e(X,X)$. The following facts of $G(n,p)$ are easily derived from Lemma 7.

Fact 8. *For all $p\colon \mathbb{N} \to (0,1)$, the following properties hold a.a.s. for $G = G(n,p)$.*

(i) For every $X \subseteq V(G)$, we have

$$\left| e(X) - \binom{|X|}{2}p \right| < \sqrt{n^2 + 6p(1-p)\binom{|X|}{2}n} + n < 4n^{3/2};$$

(ii) For every $X, Y \subseteq V(G)$, we have

$$\Big| e(X,Y) - |X||Y|p \Big| < \sqrt{n^2 + 6p(1-p)|X||Y|n} + n < 4n^{3/2};$$

(iii) For every $v \in V(G)$, we have

$$\big| \deg(v) - np \big| \leq \sqrt{\log^2 n + 6p(1-p)n\log n} + \log n < 4\sqrt{n\log n}.$$

We can now present the proof of our second main result.

Proof of Lemma 3. Let v be a vertex of minimum degree in D. Let $A := N^1(v)$ and $B := N^2(v)$. We prove that $|A| \leq |B|$, which means v is a Seymour vertex. Suppose to the contrary that $|B| < |A|$. Note that by the choice of v we have $(1 - \frac{3}{2}p)|A|^2 \geq 8n^{3/2}$. Let $\vec{e}(A,B)$ denote the number of arcs oriented from A to B. By Fact 8 *(i)* we have

$$\vec{e}(A,B) = \sum_{a\in A} \deg^+(a) - e(A) \geq |A|^2 - \left(\frac{|A|^2p}{2} + 4n^{3/2}\right) = \left(1 - \frac{p}{2}\right)|A|^2 - 4n^{3/2}, \quad (3)$$

and by Fact 8 *(ii)* we have

$$\vec{e}(A,B) \leq e(A,B) \leq |A||B|p + 4n^{3/2} < |A|^2p + 4n^{3/2}. \quad (4)$$

Combining (3) and (4), we obtain $8n^{3/2} > (1 - \frac{3}{2}p)|A|^2$, a contradiction. □

To conclude, we verify Conjecture 1 for random orientations of $G(n,p)$. The next result is immediate from Lemma 3 and Fact 8 *(iii)*, noting that a.a.s. $\delta^+\big(D(n,p)\big) \geq np/8$.

Corollary 9. *If $p \in (0, 2/3)$ is a constant, then an orientation of $G = G(n,p)$ chosen uniformly at random among the $2^{e(G)}$ possible orientations of G a.a.s. has a Seymour vertex. In particular, almost every labeled oriented graph contains a Seymour vertex.*

3. Concluding remarks

In this paper we verify Seymour's Second Neighborhood Conjecture for a family of sparse random graphs. This is a first step towards confirming it on $G(n,p)$ for all p. We also show that, when $p \in (0, 2/3)$ is a constant, the further condition of a high (though sublinear) minimum outdegree is sufficient for every vertex of minimum outdegree to be a Seymour vertex. This result may be used as a tool in further proofs, and implies that almost every (labeled) oriented graph of order n has a Seymour vertex. In forthcoming work, we prove that an arbitrary orientation of $G(n,p)$ satisfies the Conjecture 1 a.a.s. whenever $\limsup p < 1/4 - \beta$, for any given $\beta > 0$.

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