On the maximum number of edges in a graph with prescribed walk-nonrepetitive chromatic number *

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Abstract. *Fix a coloring* $c\colon V(G) \to \mathbb{N}$ *of the vertices of a graph* G *and let* $W = v_1 \cdots v_{2r}$ *be a walk in* G*. We say that* W *is* repetitive *(with respect to* c*) if* $c(v_i) = c(v_{i+r})$ *for every* $i \in \{1, \ldots, r\}$*; and that* W *is* boring *if* $v_i = v_{i+r}$*, for every* $i \in \{1, \ldots, r\}$*. Finally, we say that* c *is a* walk-nonrepetitive coloring *of* G *if every repetitive walk is boring, and we denote by* $\sigma(G)$ *the* walk-nonrepetitive chromatic number*, i.e., the minimum number of colors in a walk-nonrepetitive coloring of* G*. In this paper we explore the maximum number of edges in a graph* G *with* n *vertices for which* $\sigma(G) = k$*, for* $k \geq 4$*. In [Barát and Wood 2008] it was shown that* $e(G) \leq \frac{1}{2}(k-1)n$*. We show that* $e(G) = \frac{1}{2}(k-1)n$ *if and only if* G *is a union of disjoint copies of* K_k*. We also show that this upper bound can be improved for connected graphs for the case* $k = 4$*: if* G *is a connected graph for which* $\sigma(G) = 4$ *and* $|V(G)| \geq 5$*, then* $e(G) \leq \frac{4}{3}|V(G)|$*.*

Resumo. *Fixe uma coloração* $c\colon V(G) \to \mathbb{N}$ *dos vértices de* G *e seja* $W = v_1 \cdots v_{2r}$ *um passeio em* G*. Dizemos que* W *é* repetitivo *(com respeito a* c*) se* $c(v_i) = c(v_{i+r})$ *para todo* $i \in \{1, \ldots, r\}$*; e que* W *é* entendiante *se* $v_i = v_{i+r}$ *para todo todo* $i \in \{1, \ldots, r\}$*. Finalmente, dizemos que* c *é uma* coloração passeio não-repetitiva *de* G *se todo passeio repetitivo é entendiante, e denotamos por* $\sigma(G)$ *o* número cromático passeio não-repetitivo*, i.e., o menor número de cores em uma coloração passeio não-repetitiva de* G*. Neste artigo, exploramos o número máximo de arestas em um grafo* G *com* n *vértices para o qual* $\sigma(G) = k$*, para* $k \geq 4$*. Em [Barát and Wood 2008] foi provado que* $e(G) \leq \frac{1}{2}(k-1)n$*. Nós mostramos que* $e(G) = \frac{1}{2}(k-1)n$ *se e somente se* G *é uma união de cópias disjuntas de* K_k*. Também provamos que o limitante superior pode ser melhorado para grafos conexos quando* $k = 4$*: se* G *é um grafo conexo com* $\sigma(G) = 4$ *e* $|V(G)| \geq 5$*, então* $e(G) \leq \frac{4}{3}|V(G)|$*.*

1. Introduction

All graphs in this paper are finite, undirected and simple. Given a graph G and a vertex $v \in V(G)$, we denote by $d(v)$ (resp. $N(v)$) the *degree* (resp. the set of neighbors) of v.

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A *walk* in G is a sequence $v_1 \cdots v_r$ of vertices of G for which $v_i v_{i+1} \in E(G)$ for every $i \in \{1, \ldots, r-1\}$. A walk $W = v_1 \cdots v_{2r}$ is *boring* if $v_i = v_{i+r}$ for each $i \in \{1, \ldots, r\}$. Given a coloring $c\colon V(G) \to \mathbb{N}$, we say that W is *repetitive* if $c(v_i) = c(v_{i+r})$ for each $i \in \{1, \ldots, r\}$. Finally, we say that a coloring $c\colon V(G) \to \{1, \ldots, k\}$ is *walk-nonrepetitive* if every repetitive walk is boring, and denote by $\sigma(G)$ the minimum k for which there is a walk-nonrepetitive coloring $c\colon V(G) \to \{1, \ldots, k\}$ of G, which is called *the walk-nonrepetitive chromatic number* [Barát and Wood 2008]. Nonrepetitive colorings have attracted some attention in the last years as applications of a method related to the entropy compression method [Rosenfeld 2020]. We also recommend [Wood 2021].

Note that any walk-nonrepetitive coloring c of a graph G is a proper coloring because if $uv \in E(G)$, then uv is not a boring walk, and hence $c(u) \neq c(v)$. Moreover, all the neighbors of a same vertex must have different colors. Indeed, let $v \in V(G)$ and u and w be distinct neighbors of v. Since $uvwv$ is not a boring walk, we have $c(u) \neq c(w)$. This implies that $\sigma(G) \geq \Delta(G) + 1$, where $\Delta(G)$ denotes that maximum degree of G, and hence $e(G) \leq \frac{1}{2}\Delta(G)|V(G)| \leq \frac{\sigma(G)-1}{2}|V(G)|$, which is the upper bound proved in [Barát and Wood 2008]. This upper bound is tight because if G is a union of disjoint copies of K_k, then $\sigma(G) = k$ and $e(G) = \frac{k-1}{2}|V(G)|$. We show that such an extremal graph is unique when k divides $|V(G)|$ (Theorem 1), and that a smaller upper bound holds for connected graphs in the case $k = 4$ (Theorem 2).

2. Main results

Let G be a connected graph on n vertices, and note that if $e(G) = \frac{1}{2}\Delta(G)n$, then G is *regular*, i.e., all of its vertices have the same degree. The following result implies that, given $k \geq 4$, there is a unique graph with walk-nonrepetitive chromatic number k and such a number of edges when k divides n.

Theorem 1. *Let $k \geq 4$. If G is a $(k-1)$-regular graph for which $\sigma(G) = k$, then G is the union disjoint of copies of K_k.*

Proof. The proof follows by induction on k. First, suppose that $k = 4$, and let G be a 3-regular graph for which $\sigma(G) = 4$. Let C_1, C_2, C_3, C_4 be the color classes of a minimum walk-nonrepetitive coloring of G. As mentioned above, since $\sigma(G) = 4$, and G is 3-regular, each vertex in C_i has exactly one neighbor in each C_j with $j \in \{1, 2, 3, 4\}$ and $j \neq i$. Therefore, $H = G \setminus C_1$ is a 2-regular graph, i.e., a set of disjoint cycles. In fact, since each vertex in H has precisely two neighbors of different colors, one may prove that each component of H is a cycle of length divisible by 3. Note that if one such component C has length at least 6, then each sequence of six consecutive vertices in C forms a repetitive path that is not boring. Therefore, each component of H is a triangle.

Now, we prove that each component of G is a copy of K_4. Let $v_1 \in C_1$ and let v_2 and v_3 be distinct neighbors of v_1. Suppose, for a contradiction, that v_2 and v_3 are in distinct components of H, say H_2 and H_3, respectively. Suppose, without loss of generality, that $v_2 \in C_2$ and $v_3 \in C_3$, and let v_2' (resp. v_3') be a neighbor of v_3 (resp. v_2) in C_2 (resp. C_3). Note that v_2' and v_3 are in H_3 while v_3' and v_2 are in H_2, and hence $v_2' \neq v_2$ and $v_3' \neq v_3$. Now, let v_1' be a neighbor of v_3' in C_1, and note that $v_1'v_3'v_2v_1v_3v_2'$ is a repetitive path that is not boring, a contradiction. Therefore all of the neighbors of v_1 are in the same component of H, and hence H is a union of disjoint copies of K_4.

Now, suppose $k > 4$. Analogously to the case above, $H = G \setminus C_1$ is a $(k-2)$-regular graph for which $\sigma(H) = k - 1$. By the induction hypothesis, H is a union of disjoint copies of K_{k-1}, and analogously to the case above, all the neighbors of a vertex in C_1 must be in the same component, as desired. □

Theorem 1 implies that any connected graph G for which $|V(G)| \geq 5$ and $\sigma(G) = 4$ has less than $\frac{3}{2}|V(G)|$ edges. The next result slightly improves this upper bound.

Theorem 2. *If G is a connected graph on $n \geq 5$ vertices for which $\sigma(G) = 4$, then $e(G) \leq 4n/3$.*

Proof. The proof follows by contradiction. Let G be a connected graph with $n \geq 5$ vertices and $\sigma(G) = 4$, and suppose that $e(G) > 4n/3$. Fix a minimum walk-nonrepetitive coloring c of G. The first step of the proof is to show that G does not have some special subgraphs. For example, G has no copies of K_4. Indeed, since $\Delta(G) \leq 3$, if G has a copy K of K_4, then K is a component of G, and hence $G = K$, a contradiction to $n > 4$.

In what follows, let V_3 be the set of vertices of G with degree exactly 3, let $V_{\leq 2} = V(G) \setminus V_3$, and set $G_3 = G[V_3]$. Now, let Y be the tree obtained from the star with three edges by subdividing one of its edges (see Figure 1). We claim that G_3 has no copies of Y as a subgraph. Indeed, suppose that G_3 has a copy H of Y on the vertices u_1, u_2, u_3, u_4, w, where $N_H(u_1) = \{u_2, u_3, u_4\}$ and $N_H(u_4) = \{u_1, w\}$. Then u_2, u_3 and w are leaves of H. We may suppose, without loss of generality, that $c(u_i) = i$, for every $i \in \{1, 2, 3, 4\}$, and hence $c(w) = 2$ or $c(w) = 3$. Say $c(w) = 2$. The case $c(w) = 3$ follows analogously. As $w, u_2 \in V_3$, we have $d(w) = d(u_2) = 3$, and hence w has a neighbor v_1 for which $c(v_1) = 1$, and u_2 has a neighbor v_4 for which $c(v_4) = 4$. Note that $v_1 \neq u_1$, otherwise $d(u_1) \geq 4$. Then $v_1 w u_4 u_1 u_2 v_4$ is a repetitive walk that is not boring, a contradiction.

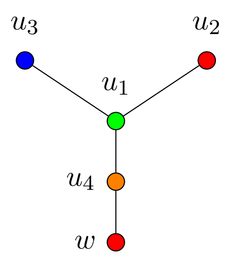

Figure 1. The Y graph.

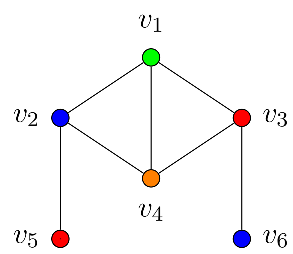

Figure 2. The R graph.

Now, let R be graph obtained from a K_4 by removing an edge $v_2 v_3$, and adding two new vertices, each adjacent to one of v_2 and v_3 (see Figure 2). We claim that G has no copies of R as a subgraph. Indeed, suppose that G has a copy H of R on the vertices $v_1, v_2, v_3, v_4, v_5, v_6$, where $N_H(v_5) = \{v_2\}$, $N_H(v_6) = \{v_3\}$. We may suppose that $c(v_i) = i$, for every $i \in \{1, 2, 3, 4\}$, and hence $c(v_5) = c(v_3) = 3$ and $c(v_6) = c(v_2) = 2$. If $d_G(v_5) = d_G(v_6) = 1$, then $G = H$, and hence $e(G) = 7 < 8 = 24/3 = 4n/3$, a contradiction. Then either v_5 or v_6, say v_5, has a neighbor, say w, outside H. Since c is a walk-nonrepetitive coloring, $c(w) \notin \{c(v_2), c(v_5)\} = \{2, 3\}$, and hence $c(w) = i \in \{1, 4\}$. Then $w v_5 v_2 v_i v_3 v_6$ is a repetitive walk that is not boring, a contradiction.

Given two disjoint sets of vertices U and W, we denote by $e(U, W)$ the number of edges joining vertices of U to vertices of W. Now, let $\mathcal{H}$ be the set of components of G_3, and given $H \in \mathcal{H}$, let $e_H = e(V(H), V_{\leq 2})$.

Claim 1. *For every component H of G_3 we have $|V(H)| \leq e_H$.*

Proof. If $\Delta(H) \leq 2$, then each vertex of H has a neighbor in $V_{\leq 2}$ and the claim follows. Thus, we may assume that H has a vertex u for which $d_H(u) = 3$. Let v_1, v_2, v_3 be the neighbors of u. If v_i has a neighbor w in $V_3 \setminus \{u, v_1, v_2, v_3\}$, then $\{u, v_1, v_2, v_3, w\}$ induces a copy of Y in G_3, a contradiction. Thus H has only four vertices and hence $e_H = 6 - 2e(G[\{v_1, v_2, v_3\}])$. If $e(G[\{v_1, v_2, v_3\}]) \leq 1$, then the claim follows, and if $e(G[\{v_1, v_2, v_3\}]) = 3$, then H is a copy of K_4, which is a contradiction. Then $e(G[\{v_1, v_2, v_3\}]) = 2$, and H is the graph obtained from K_4 by removing one edge. Suppose that $v_1v_2 \notin E(H)$ and let w_1 and w_2 be the remaining neighbors of v_1 and v_2, respectively. Since G does not have a copy of R, we have $w_1 = w_2$. Note that $c(w_1) \notin \{c(v_1), c(v_2)\}$ because c is a proper coloring. Then $c(w_1) \in \{c(u), c(v_3)\}$, and hence v_1 has two neighbors with the same color, a contradiction. $\square$

By Claim 1, we have $|V_3| = \sum_{H \in \mathcal{H}} |V(H)| \leq \sum_{H \in \mathcal{H}} e_H = e(V_3, V_{\leq 2})$. Moreover, $e(V_3, V_{\leq 2}) \leq 2|V_{\leq 2}|$, because $d(v) \leq 2$ for every $v \in V_{\leq 2}$. Therefore, $|V_3| \leq 2|V_{\leq 2}|$. Since $|V_3| + |V_{\leq 2}| = n$, we have $|V_3| \leq 2n/3$. Therefore,

$$2e(G) = \sum_{v \in V_3} d(v) + \sum_{v \in V_{\leq 2}} d(v) \leq 3|V_3| + 2|V_{\leq 2}| = 2n + |V_3| \leq 2n + \frac{2n}{3} = \frac{8n}{3}$$

as desired. $\square$

3. Future work

In this paper we have shown the uniqueness of the extremal graph with walk-nonrepetitive chromatic number k and order divisible by k for any fixed $k \geq 4$, which is a disconnected graph; and we also presented a smaller upper bound on the maximum number of edges in a connected graph G with walk-nonrepetitive chromatic number 4. So far we have not found a graph with a matching number of edges, so this upper bound may still be improved. We were also able to show that, for general $k \geq 4$, G does not contain generalization of the Y graph, but a stronger structure seems to be needed to generalize this bound. These results may be later combined to characterize the extremal graphs with any order.

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