Proper edge colorings of complete graphs without repeated triangles *

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Abstract. *In this paper, we consider the problem of computing the minimum number of colors needed to properly color the edges of a complete graph on n vertices so that there are no pair of vertex-disjoint triangles colored with the same colors. This problem was introduced recently (in a more general context) by Conlon and Tyomkyn, and the corresponding value was known for odd n. We compute this number for another infinite set of values of n, and discuss some small cases.*

1. Introduction

In this paper, all graphs considered are finite and simple (no multiple edges or loops allowed). Let G be a graph, we say that an edge coloring $c\colon E(G) \to \mathbb{N}$ is *proper* if for every two distinct edges $e_1, e_2 \in E(G)$, we have $c(e_1) \neq c(e_2)$ whenever $e_1 \cap e_2 \neq \varnothing$. A recent paper [Conlon and Tyomkyn 2020] introduced a new restriction on edge colorings, as follows: let $n, k \geq 2$ be integers, and H be a fixed graph. The number $f_k(n, H)$ is defined as the smallest C with the property that there is a proper edge coloring of K_n with C colors containing no k vertex-disjoint color-isomorphic copies of H (i.e., isomorphic

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copies of H for which there is an isomorphism between them that preserves the colors of the edges).

In this paper, we deal with the case $k = 2$ and $H = K_3$. Conlon and Tyomkyn show that $f_2(n, K_3) = n$ whenever n is odd, and that this implies that $f_2(n, K_3) \leq n + 1$ for even n. Furthermore, as any proper coloring of K_n requires at least $n - 1$ colors, it follows that $f_2(n, K_3) \geq n - 1$. Hence, the remaining problem is to decide, for each even n, whether $f_2(n, K_3)$ equals to $n - 1$, n or $n + 1$. In Section 2, we deal with the cases $n \in \{4, 6, 8, 12\}$, determining $f_2(n, K_3)$ (with the aid of a computer in the case $n = 12$). In Section 3, we give an infinite set of integers n for which $f(n) = n - 1$. Finally, Section 4 is devoted to some concluding remarks and suggestions of future work.

2. Small cases

In this section, we determine $f_2(n, K_3)$ for $n \in \{4, 6, 8, 12\}$. It is clear that $f_2(4, K_3) = 3$, since there are no pair of vertex-disjoint triangles in K_4, and hence we are simply looking for any proper edge coloring. Perhaps surprisingly, the next two results show that, for both $n = 6$ and $n = 8$, $n + 1$ colors are needed.

In the proofs of this section, we say that two vertex-disjoint triangles are *similar* in an edge coloring of a graph if their edges get the same three colors; and say that a color is *full* if the edges that receive it form a perfect matching in the graph.

Theorem 1. $f_2(6, K_3) = 7$.

Proof. By the previous considerations, it is enough to prove that every edge-coloring of K_6 with at most six colors contains a pair of vertex-disjoint similar triangles. Assume that this is not the case, and let $V(K_6) = \{1, \ldots, 6\}$. As K_6 has fifteen edges, and each of the at most six color classes consists of at most three edges, it follows that at least three colors are full. Furthermore, the union of any two perfect matchings induces a cycle of length 6. We may assume without loss of generality that the cycle 123456 is colored alternately with colors 1 and 2. We will call an edge a *short chord* if it connects two vertices within distance 2 in this cycle (in particular, a short chord forms a triangle with edges of color 1 and 2). It is easy to see that if we have three short chords with the same color, we get two distinct similar triangles. Furthermore, two short chords of the same color must be of the form $i(i + 2)$ and $(i + 1)(i + 3)$ for some i (indices mod 6), since otherwise they would generate two disjoint similar triangles.

Suppose that there is some short chord colored with a full color, say 13 is colored with color 3. It is simple to check that the remaining edges of color 3 must be 25 and 46, which is a contradiction, since 123 and 456 would be similar triangles.

This implies that no short chord belongs to a full color. As there are 6 short chords and each of the (at most three) colors that are not full can cover at most two short chords, we conclude that every color which is not full appears in exactly two short chords. Without loss of generality, we may assume that 13 and 24 got color 4, 35 and 46 got color 5, and 51 and 62 got color 5. In this case, the triangles 135 and 246 are similar, a contradiction. □

Theorem 2. $f_2(8, K_3) = 9$.

Proof. By the previous considerations, it is enough to prove that every edge-coloring of K_8 with at most 8 colors contains a pair of vertex-disjoint similar triangles. Assume that this is not the case, and let $\{1, \dots, 8\}$ be the vertex set of the K_8. As K_8 has 28 edges, and each of the at most eight color classes consists of at most four edges, it follows that at least four colors are full. Let us split the proof into two cases.

Case 1: There are two full colors whose union is a C_8. Without loss of generality, we may say that the colors 1 and 2 span the cycle 12345678 (in a way that 12 is colored with color 1).

If a short chord, say, 82, belongs to a full color (say, 3), then it is simple to check that the other edges of color 3 must be 15, 36 and 47, or otherwise there will be two disjoint triangles colored with $1, 2$ and 3. Now consider the short chord 46, which is colored, say, with color 4. Either the chord 35 or 57 is colored with a new color, say, 5. We will assume that 57 is colored with 5. Now, it is readily checked that there is no edge incident on vertex 2 colored with color 4 (otherwise there would be two disjoint triangles colored with $1, 3$ and 4). As we have at most eight colors, we conclude that every color except 4 must appear in some edge incident to vertex 2. In particular, some edge must be colored with color 5, and again it is simple to check that this is a contradiction, for it would create either two disjoint triangles colored with $1, 2$ and 5 or $2, 3$ and 5.

Therefore, we may assume that every short chord belongs to a color that is not full. As we have at most four such colors and eight short chords, we conclude that there are exactly four colors that are not full (and hence, each such color covers exactly three edges), and each of them covers exactly two short chords that must be neighboring. That is, we may assume that the edges 13 and 24, 35 and 46, 57 and 68, 71 and 82 are colored with color 3, 4, 5 and 6, respectively. The third edge of color 3 must be 58 (as the other edges joining vertices from $\{5, 6, 7, 8\}$ are already colored with other colors), and, similarly, 72, 14 and 36 must be colored with color 4, 5 and 6, respectively. The triangles 134 and 568 are similar (both have color set 135), a contradiction.

Case 2: Every pair of full colors spans two copies of C_4.

In this case, it is simple to check that the union of three full colors, say, 1, 2 and 3, must span a cube such that each color class corresponds to one dimension. A fourth full color, say, 4, cannot cover any edge inside a face of the cube, otherwise it would create a similar triangle in the opposite face. This implies that the edges of color 4 are precisely the main diagonals of the cube.

The colors 5, 6, 7, 8 must cover the diagonal of the faces of the cube. It is clear that, inside a face, the two diagonals must be of different colors (otherwise the opposite face would have a similar triangle). Furthermore, if one face has diagonals colored with 5 and 6, the opposite must have the other two colors, i.e., 7 and 8, not to create a similar triangle. Moreover, two adjacent faces cannot have the same pair of colors in their diagonals, since it would generate two similar triangles together with the main diagonals of the cube.

The observations above imply that each of the $\binom{4}{2} = 6$ pairs of the colors $5, 6, 7, 8$ appears in the diagonals of exactly one of the faces of the cube. Consider the three faces that contains a diagonal of color 5. No two of these faces can be opposite, and hence the three are mutually adjacent. This is a contradiction, since it is checked that there are no three pairwise independent diagonals in three mutually adjacent faces of the cube. $\square$

Finally, with aid of a computer, we found that $f_2(12, K_3) = 12$ (see Figure 1), which is the smallest even value of n such that $f_2(n, K_3) = n$. This shows that each value of $\{n-1, n, n+1\}$ can occur as $f_2(n, K_3)$ for some even n. The technique used was a reduction to a SAT problem, which was generated using Sage [The Sage Developers 2020] and solved using a SAT solver [Biere 2018]. The code and the coloring are available at `https://github.com/robertoparente/k3copy_sat`.

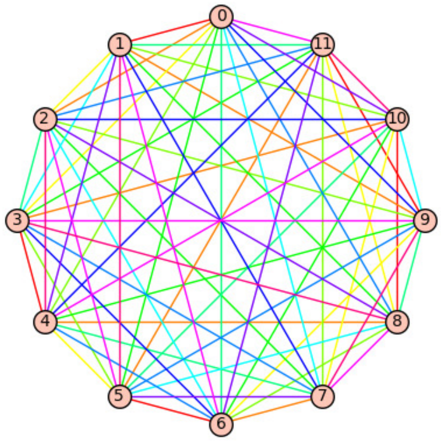

Figure 1. A coloring of K_{12} with 12 colors

3. An infinite family of values of n

In this section, we compute $f_2(n, K_3)$ for the first known infinite family of even n.

Theorem 3. *If $n = 3^t + 1$, where t is a positive integer, then $f_2(n, K_3) = n - 1$.*

Proof (sketch): Suppose that $n = 3^t + 1$, and let u be a fixed vertex of K_n. We associate each of the remaining 3^t vertices to a distinct element of $\mathbb{Z}_3^t = \{x = (x_1, \ldots, x_t) \colon x_i \in \mathbb{Z}_3, \ \forall i \in [t]\}$.

We define the coloring $\chi \colon E(K_n) \to \{0, 1, 2\}^t = \mathbb{Z}_3^t$ as follows:

$$\chi(\{x, y\}) = \begin{cases} (2x_1, \ldots, 2x_t), & \text{if } y = u; \\ (x_1 + y_1, \ldots, x_t + y_t), & \text{if } u \notin \{x, y\}, \end{cases}$$

where the addition is considered modulo 3. It is simple to check that χ is a coloring with the desired properties. $\square$

4. Concluding remarks

In this paper, we considered proper edge coloring in which vertex-disjoint triangles receive different sets of colors. As mentioned before, this problem was posed in a broader context of forbidding any number of repetitions of any fixed graph in proper colorings of K_n. We plan to investigate the problem for other number of copies and other fixed graphs.

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