A generalization of the block decomposition for k-connected graphs*

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Abstract. *The decomposition of a connected graph by the set of its cut-vertices, sometimes called the "block decomposition" or "block tree" of a graph, is a well known and basic concept in graph theory. This decomposition, however, does not provide meaningful information when applied to a k-connected graph for $k \geq 2$. There has been a number of attempts to generalize the construction of the block decomposition of a graph for the case of k-connected graphs. In this work, we present an outline of one such attempt by Karpov. We also present some applications of this decomposition to the study of planarity, the chromatic number, and critically 2-connected graphs.*

A basic concept defined for connected graphs is its decomposition by the set of its cut-vertices, or cut-points, the so-called block decomposition of a connected graph. A block of a connected graph G is defined as a maximal subset of vertices that induces either a 2-connected graph or the connected graph on two vertices, the graph K_2. The blocks of the decomposition of G form a tree-like structure. More formally, consider the bipartite graph where one part is the set of blocks of G and the other part is the set of cut-vertices of G, and a block B is adjacent to a cut-vertex u if and only if $u \in B$. It is a classic result that the above graph is a tree. Such a decomposition is a very basic tool in graph theory. Note however that any 2-connected graph has no cut-vertices and has only one block, which consists of the whole set of vertices of the graph. Thus, such a decomposition is trivial in that case. Some attempts to extend the block decomposition to the case of k-connected graphs include defining a block as a maximal set of vertices that induce a $(k+1)$-connected graph. This definition, although simple, does not share many properties in common with the classic block decomposition of a connected graph. Other works attempt to define the blocks of a k-connected graph determined by a k-cutset S in the following way. One considers the sets B_i which are the union of the set S with the vertex-sets of the components of the graph $G - S$. A block of the graph G is certain subset of one such B_i. Also, in order to preserve k-connectivity, the set S is converted into a clique in the graph $G[B_i]$. One can proceed recursively in this way by choosing another k-cutset of the new graph $G[B_i]$ and finding smaller subgraphs to the point in which all subgraphs no longer contain k-cutsets. Note that this procedure produces the classic block decomposition when the cutsets are the cut-vertices of a connected graph, and such a decomposition is unique. However, when $k \geq 2$, the order in which the k-cutsets are considered does matter, i.e., different orders may produce different decompositions, e.g., when some cutsets separate other cutsets. Tutte constructed a tree that describes the mutual arrangement of the 2-cutsets of a 2-connected graph [Tutte 1966].

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We will present another approach developed by Karpov to define the decomposition by a set of pairwise independent cutsets, i.e, cutsets that do not separate each other in the graph [Karpov 2013]. We independently proved many of the facts presented in this paper.

In what follows, we define the decomposition of a k-connected graph by an arbitrary set of its cutsets. Then, we discuss some properties of this decomposition when applied using a set of pairwise independent cutsets. We then show a special case of this decomposition for 2-connected graphs. Finally, we show some applications of the decomposition to the study of planarity, the chromatic number, the choice number, and critically 2-connected graphs.

1. Decomposing a k-connected graph by a set of cutsets

Let $G = (V, E)$ be a connected graph. A set $R \subset V$ is called *cutset* if the graph $G - R$ is disconnected. If the cardinality of R is k, we also call R a k-cutset. If $k = 1$, we call the vertex contained in R a *cut-vertex*. In this case, the following definitions hold analogously for such a vertex. For an integer $k \geq 1$, the graph G is said to be *k-connected* if $|V| > k$ and G has no cutset with fewer than k vertices. A set $R \subset V$ *separates a set $X \subset V$ in G* if there are two vertices of $X \setminus R$ that belong to different components in the graph $G - R$. If the graph G is clear from the context, we simply say that R separates X. Let $R \subset V$, and let $X, Y \subset V$. The set R *separates X and Y in G* if no component of the graph $G - R$ contains vertices of both X and Y. This definition holds analogously if X or Y are vertices.

Let $k \geq 1$ be fixed. Let G be a k-connected graph. Let $\mathrm{Cut}(G)$ and $\mathrm{Cut}_r(G)$ be the set of all cutsets and r-cutsets of G respectively. Let $\mathfrak{S} \subset \mathrm{Cut}_k(G)$ be an arbitrary set of k-vertex cutsets of G. The decomposition of G by $\mathfrak{S}$ is defined as follows.

A set of vertices $A \subset V(G)$ is an *$\mathfrak{S}$-nonseparable set of G* if no cutset of $\mathfrak{S}$ separates A in G. Observe that the empty set is also a $\mathfrak{S}$-nonseparable set of G. Let $\mathrm{Ns}(G; \mathfrak{S})$ be the set of all non-empty $\mathfrak{S}$-nonseparable sets of G. A set of vertices $A \subset V$ is an *$\mathfrak{S}$-block of G* if A is an inclusion-maximal element of $\mathrm{Ns}(G; \mathfrak{S})$. When the set $\mathfrak{S}$, is clear from the context, we simply call those elements *blocks of G*, or *blocks*.

Some properties of this decomposition are the following. First, it is clear from the definition that every vertex is contained in some block of G. Also, the graph contains at least one block and no block is empty since every member of $\mathrm{Ns}(G; \mathfrak{S})$ is non-empty.

For a block A, we define the *boundary* of A as the set of vertices of A that are contained in some member of $\mathfrak{S}$. We also define the *interior* of A as the complement of the boundary with respect to A. It is easy to show that, for any two different blocks of G with nonempty intersection, such an intersection is contained in some member of $\mathfrak{S}$. This fact implies that the interiors of two distinct blocks are disjoint. Thus, any vertex of G is either a member of the interior of a unique block, or is a member of at least one boundary.

Let $S \in \mathrm{Cut}_k(G)$ be a cutset of G. Consider as an example the decomposition of G by the set $\mathfrak{S} = \{S\}$. It is easy to see that the $\mathfrak{S}$-blocks of G are the union of the set S and the components of the graph $G - S$. The number of such blocks is the number of components of $G - S$. The interiors of the blocks are the vertex-sets of such components.

One can also characterize the boundary of a block A by showing that this set is the set of all vertices of A that are adjacent to a vertex of $V(G) \setminus A$ in G. This tells us that the boundary of a block is a set that separates the interior of the block and the rest of the graph. Thus, if the interior is non-empty, then by the k-connectivity of G, every block of G has at least $k+1$ vertices.

These are some of the basic properties of the decomposition of a k-connected graph by $\mathfrak{S}$. In the next section, we assume more properties about the set $\mathfrak{S}$ in order to gain more structure from this decomposition.

2. Decomposing a k-connected graph by a set of pairwise independent cutsets

In the decomposition defined above, it is possible that some k-cutsets of $\mathfrak{S}$ separate other cutsets of the set. We call a pair of k-cutsets *independent* if none of them separates the other in G. If we limit ourselves to studying the decomposition of a k-connected graph G by a set of pairwise independent cutsets, then the tree-like structure found in the classic block decomposition of a connected graph can be established. Using a set of pairwise independent cutsets could seem like a strong condition to impose. However, one can easily show that for a pair S and T of k-cutsets of G, either S and T are independent, or each of them separates the other in G.

From this point on, we let $\mathfrak{S}$ be a set of pairwise independent k-cutsets of a k-connected graph G. An initial observation is that, in this case, every cutset of $\mathfrak{S}$ is contained in a block of G, given that no other cutset of $\mathfrak{S}$ can separate it. In order to study some other properties of the decomposition of G by $\mathfrak{S}$, it is useful to construct a new graph G' by taking a copy of the graph G and turning each set of $\mathfrak{S}$ into a clique in G' by adding the remaining edges between vertices of every member of $\mathfrak{S}$. One can first show that every cutset of $\mathfrak{S}$ is also a cutset of the graph G', and pairs of cutsets of $\mathfrak{S}$ are also independent in the new graph G'. Furthermore, the blocks of the decomposition of G' by $\mathfrak{S}$ are exactly the blocks of the decomposition of G by $\mathfrak{S}$. This construction allows one to show that, if $\mathfrak{S}$ is non-empty, then every block of G not only contains a cutset, but also contains it properly. For any block A, it is possible to see that every cutset of $G'[A]$ is also a cutset of G. Hence, the graph $G'[A]$ is also k-connected.

Let $\mathrm{T}(G;\mathfrak{S})$ be the bipartite graph in which one part is the set $\mathfrak{S}$, the other part is the set of $\mathfrak{S}$-blocks of G, and a cutset of $\mathfrak{S}$ is adjacent to a block of G if and only if the block contains the cutset. An important result of this theory is that *the graph* $\mathrm{T}(G;\mathfrak{S})$ *is a tree* and that *the degree of a cutset of* $\mathfrak{S}$ *in* $\mathrm{T}(G;\mathfrak{S})$ *is at least* 2, i.e., *every leaf of* $\mathrm{T}(G;\mathfrak{S})$ *is a block of* G. Also, a set $R \in V(\mathrm{T}(G;\mathfrak{S}))$ separates two other sets $X, Y \in V(\mathrm{T}(G;\mathfrak{S}))$ in G if and only if the vertex R separates the vertices X and Y in $\mathrm{T}(G;\mathfrak{S})$. These properties are similar to the properties of the block decomposition of a connected graph. In fact, both decompositions coincide when $k=1$ in the decomposition described above.

3. The case of $k=2$

We now show some of the structure of the above decomposition when applied to a 2-connected graph. We call a k-cutset of G *isolated* if it is independent from all other k-cutsets of G.

Let G be a 2-connected graph. We now let $\mathfrak{S}$ be not only a set of pairwise independent 2-cutsets, but the set of all isolated 2-cutsets of G. We again let G' be a copy of the graph G, in which every member of $\mathfrak{S}$ was completed to a clique in G'. A key observation of the decomposition, in this case, is that the 2-cutsets of G that are not isolated but are contained in a block A of G are also 2-cutsets of the graph $G'[A]$ and are not isolated in $G'[A]$ either. In fact, if we consider a 2-cutset S of G that is not isolated, the decomposition of G by the set $\mathfrak{S} = \{S\}$ consists of only two blocks A and B, and the graphs $G[A]$ and $G[B]$ are not 2-connected. This allows one to prove that every 2-connected graph G with no isolated cutset is *either* 3*-connected or a cycle*. Also, a direct consequence of this fact is that for every block A of the decomposition of a 2-connected graph G by the set of its isolated cutsets $\mathfrak{S}$, the *graph* $G'[A]$ *is either* 3*-connected or a cycle*. In the latter case, every pair of vertices that are non-adjacent in the cycle form a 2-cutset of G that is not isolated. *Every non-isolated* 2*-cutset of* G *is of this form.*

4. Some applications

Let G be a 2-connected graph, and let $\mathfrak{S}$ be the set of its isolated 2-cutsets. With the help of the decomposition defined above, one can show that G is planar if and only if for every block A of G, the graph $G'[A]$ is planar. This result is similar to a planarity characterization proven by Mac Lane, who studied the decomposition of a 2-connected graph into *atoms* and proved that a 2-connected graph G is planar if and only if all of its atoms are planar [Mac Lane 1937]. In this work, the atoms are in fact subdivisions of $G'[A]$.

Let G be a k-connected graph, and let $\mathfrak{S}$ be the set of its isolated k-cutsets. It is trivial to see that the chromatic number of G is at most the chromatic number of the graph G'. However one can show that the chromatic number of the graph G' is the maximum chromatic number of the graphs $G'[A]$, where A is a block of G. It is also possible to give an upper bound for the chromatic number of G in terms of the graphs $G[A]$. Let A be a block of G such that $\chi(G[A])$ is maximum. Then $\chi(G) \leq k - 1 + \chi(G[A])$. A similar inequality holds for list colorings. If we choose A as above for the choice number, then $\mathrm{ch}(G) \leq k + \mathrm{ch}(G[A])$.

A *critically* k-connected graph is a k-connected graph G of order at least $k + 2$ such that for any vertex v of G the graph $G - v$ is not k-connected. It was proved in [Hamidoune 1980] that a critically k-connected graph has at least two vertices of degree less than $(3k-1)/2$. One can use the block decomposition of a 2-connected graph to show that every critically 2-connected graph has at least 4 vertices of degree 2.

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