On the Partial Grundy Number of a Graph Minus a Matching

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Abstract. *Given a k-coloring $S_1, \ldots, S_k$ of G, a vertex in S_i is said to be greedy if it has neighbors in S_j, for every $j < i$. Thus, a Grundy coloring can be seen as a coloring where every vertex is greedy. In contrast, a partial Grundy coloring is defined as a coloring in which each color class has at least one greedy vertex; the maximum number of colors in a partial Grundy coloring is denoted by $\partial\Gamma(G)$. In this paper, we investigate some conjectures about Grundy colorings, already known not to hold, adapted to partial Grundy colorings. We prove that, while two of them also do not hold, a third does. Namely, that given a graph G and a matching M of G, we have that $\partial\Gamma(G) \leq 2\partial\Gamma(G - M)$.*

1. Introduction

The definitions and notations used in this article are standard and can be found in any graph theory book. Also, we consider only simple graphs. Given two graphs G and H, the Cartesian product $G\square H$ is the graph with vertex set $V(G\square H) = V(G) \times V(H)$ and the following edge set:

$$E(G\square H) = \{(a, x)(b, y) | (a = b \text{ and } xy \in E(H)), \text{ or } (ab \in E(G) \text{ and } x = y)\}.$$

An *independent set* in a graph G is a subset $S \subseteq V(G)$ such that $uv \notin E(G)$ for every $u, v \in S$. A k*-coloring* of G is a partition $\mathcal{P}$ of $V(G)$ into k sets and is *proper* if S is an independent set for each $S \in \mathcal{P}$. The *chromatic number* $\chi(G)$ *of* G is the minimum value k for which G has a proper k-coloring. From now on, a proper coloring will be called simply a coloring.

Graph coloring problems are perhaps the most studied problems in graph theory, due to their practical and theoretical importance. It is one of the 21 Karp's NP-complete problems [Karp 1972], and deciding $\chi(G) \leq k$ remains hard even for $k \geq 3$ fixed. Given the difficulty of this problem, an alternative is to apply heuristics to solve it. Many of these heuristics give rise to parameters that study their worst case; among them, we are interested in studying a variation of the greedy heuristic.

Given a k-coloring $\mathcal{P}$ of G, a function $f : \mathcal{P} \to \{1, \ldots, k\}$ is called a k*-labeling* of $\mathcal{P}$. We call an element $C \in \mathcal{P}$ a *color class*, and the value $f(C)$ is called *color (of C in f)*. Given $C \in \mathcal{P}$ and $u \in C$, we say that u is a *greedy vertex* in f if $N(u) \cap C' \neq \emptyset$ for all $C' \in \mathcal{P}$ such that $f(C') < f(C)$. We make an abuse of notation and write $f(u)$ to denote $f(C)$, where $u \in C$. If all vertices of G are greedy in f, we say that f is a *Grundy coloring*, and we say that f is a *partial Grundy coloring* if each C contains at least one greedy vertex for every $C \in \mathcal{P}$. The largest k for which G has a Grundy coloring is called *Grundy number of* G and is denoted by $\Gamma(G)$, while the maximum k for which G has a partial Grundy coloring is called the *partial Grundy number of* G and is denoted by $\partial\Gamma(G)$. These parameters were defined in [Christen and Selkow 1979] and [Erdős et al. 2003], respectively, and, unlike Grundy colorings, much can still be learned about partial Grundy colorings. In this paper, we investigate the adaptation to partial Grundy colorings of three conjectures, now known not to hold, about the Grundy number of graphs, in particular of the Cartesian product of graphs.

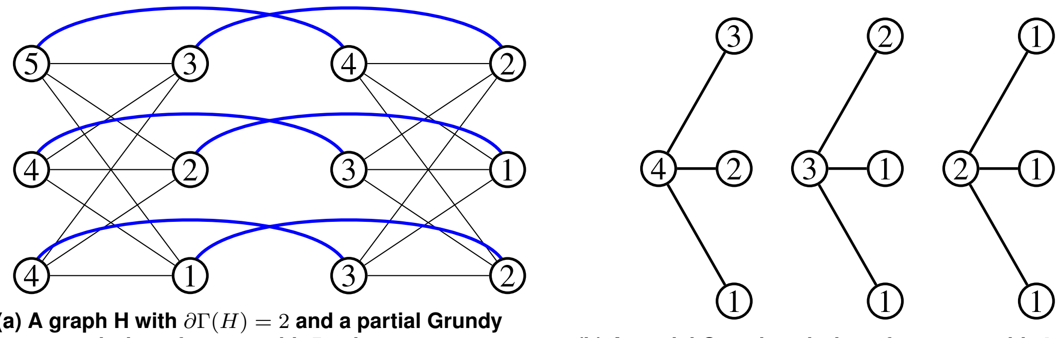
The first conjecture of interest is the following, proved not to hold in [Campos et al. 2012] by the construction of a counterexample of a graph H such that $\Gamma(H) = 3$ and $\Gamma(K_2 \Box H) = 7$.

Conjecture 1 ([Asté et al. 2010], refuted in [Campos et al. 2012]). *For every pair of graphs G and H, it holds that $\Gamma(G \Box H) \leq (\Delta(G) + 1)\Gamma(H)$.*

Conjecture 1 had actually been proposed as a generalization of the following conjecture. Hence the example given in [Campos et al. 2012] refutes also this conjecture.

Conjecture 2 ([Balogh et al. 2008], refuted in [Campos et al. 2012]). *For any graph H, $\Gamma(K_2 \Box H) \leq 2\Gamma(H)$.*

As done in [Campos et al. 2012], we refute the analogue of both conjectures by presenting a graph H that satisfies $\partial\Gamma(H) = 2$ and $\partial\Gamma(K_2 \Box H) = 5$ (see Figure 1a).



(a) A graph H with $\partial\Gamma(H) = 2$ and a partial Grundy coloring of $K_2 \Box H$ with 5 colors.

(b) A partial Grundy coloring of $S_3 \Box K_{1,3}$ with 4 colors.

Additionally, in the following theorem we give an infinite class of graphs that do not satisfy the analogue of Conjecture 1. Given a positive integer n, let S_n denote the graph with no edges and n vertices. For $n \geq 2$, the theorem below implies that $\partial\Gamma(S_n \Box K_{1,n}) \geq n + 1 > 2 = (\Delta(S_n) + 1)\partial\Gamma(K_{1,n})$, thus disproving the analogue of Conjecture 1.

Theorem 1. *Let $n \geq 2$. Then, $\partial\Gamma(S_n) = 1$, $\partial\Gamma(K_{1,n}) = 2$ and $\partial\Gamma(S_n \Box K_{1,n}) = n + 1$.*

The third and last conjecture of interest is the one below. The credits are given to Havet and Zhu in [Campos et al. 2012], but without an explicit citation.

Conjecture 3 (Havet and Zhu, refuted in [Campos et al. 2012]). *If G is any graph and M is a matching in G, then $\Gamma(G) \leq 2\Gamma(G - M)$.*

Observe that Conjecture 3 implies Conjecture 2. This is because if Conjecture 3 is valid, taking $G = K_2 \Box H$ and M the matching such that $G - M = S_2 \Box H$, it follows that $\Gamma(K_2 \Box H) \leq 2\Gamma(S_2 \Box H)$. As it is known that $\Gamma(S_2 \Box H) = \Gamma(H)$, the result follows.

Interestingly enough, this is not the case for partial Grundy colorings due to the fact that $\partial\Gamma(S_n \square H) \leq \partial\Gamma(H)$ does not necessarily hold, as we have seen in Theorem 1. Indeed, we prove that the analogue of Conjecture 3 holds for partial Grundy colorings, despite the fact that the analogue of Conjecture 2 does not.

Theorem 2. *Let G be a graph and M be a matching in G. Then, $\partial\Gamma(G) \leq 2\partial\Gamma(G-M)$.*

The paper is organized as follows. In Sections 2 and 3, we give outlines of the proofs of Theorems 1 and 2, respectively. In Section 4 we make our final comments.

2. Outline of the proof of Theorem 1

Note that, fixing $n \geq 1$, we have $\partial\Gamma(K_{1,n}) = 2, \partial\Gamma(S_n) = 1$, and because $\Delta(G\square H) = \Delta(G)+\Delta(H)$, we get $\partial\Gamma(S_n\square K_{1,n}) \leq \Delta(S_n\square K_{1,n})+1 = \Delta(S_n)+\Delta(K_{1,n})+1 = n+1$.

To prove that $\partial\Gamma(S_n\square K_{1,n}) \geq n+1$, we provide a partial Grundy $(n+1)$-coloring of $S_n\square K_{1,n}$. Note that $S_n\square K_{1,n}$ can be seen as a galaxy of n stars. We can then color this graph giving each center u_i of the i-th star the color $i+1$ for each $i \in \{1,\dots,n\}$ and color the neighborhood $N(u_i)$ in a way that all colors in $\{1,\dots,i\}$ are used. So each u_i is a greedy vertex and $\partial\Gamma(S_n\square K_{1,n}) = n+1$ follows. See Figure 1b for an example.

3. Outline of the proof of Theorem 2

Let f be a partial Grundy labeling of coloring $\mathcal{P}$ of G with $k = \partial\Gamma(G)$ colors. We show that there is a partial Grundy coloring of $G - M$ with at least $k/2$ colors. The general idea of the proof is to remove one edge of M at a time, making sure that not too many colors containing greedy vertices are lost; such a color is called a *greedy color*. For this, observe that, given $uv \in M$ and supposing $f(u) > f(v)$, the only case where the removal of uv can decrease the number of greedy colors is when u is the only greedy vertex in color class $f(u)$, and v is the only neighbor of u in $f(v)$. Intuitively, our proof consists in showing that such situation can happen at most $k/2$ times. In order to formalize our proof, we need further definitions.

Given $uv \in M$, with $f(u) > f(v)$, we say that u is the *leading vertex of* uv. We say that u is a *leader in color class C* if u is a greedy vertex and is a leader of some edge $uv \in M$. If $uv \in M$ with leader u is such that u is greedy and v is the only neighbor of u in $f(v)$, then we say that uv is *dangerous for* u. We say that color class C is *endangered* if every greedy vertex in C is incident to some dangerous edge. Finally, for each endangered color class C, we choose exactly one of its greedy vertices, u_C, to try to keep greedy throughout the proof. A non-endangered color class is called *saturated* if it is incident to M. We denote by $D(f,M)$ the set of endangered color classes and by $S(f,M)$ the set of saturated color classes. Also, denote by $d(f,M)$ the number of endangered color classes, and by $s(f,M)$ the number of saturated color classes. When f and M are clear from the context, we omit them from the notation. We start by proving:

Claim 1. *We can assume that every edge of M is dangerous.*

Claim 2. $s \leq k/2$.

We then prove, by induction on $d - s$, that $\partial\Gamma(G-M) \geq k - s$. The theorem follows by Claim 2. When $d - s = 0$, since the removal of each edge makes us lose exactly one color class (the endangered one), and since we have $d = s$ endangered colors, it follows that $\partial\Gamma(G-M) \geq k-d = k-s$. For $d-s>0$, the technique consists in first relabeling $\mathcal{P}$ by moving some color to the last position. By case analysis we can make sure that the difference $d-s$ decreases, so we can apply induction and obtain the desired coloring.

Let us take just one of these cases in order to give a better idea of how to proceed. Consider then that $\partial\Gamma(G-M) \geq k-s$ holds whenever $d-s \leq \ell$, and let f be a partial Grundy labeling of $\mathcal{P}$ such that $d-s=\ell+1$. By the pigeonhole principle, there must be a color class incident to 2 edges of M, say C. Suppose that C is a saturated color (i.e., it does not have a leader), and let $u_1v_1, u_2v_2 \in M$ be such that $\{v_1, v_2\} \subseteq C$. Let f' be obtained from f by relabeling every C' such that $f(C') > f(C)$ with $f(C')-1$, and relabeling C with k. Suppose now that f' is a partial Grundy coloring, i.e., after relabeling C in order to become the highest color, we get that C contains some greedy vertex. Because now u_1 and u_2 are the leaders of edges u_1v_1 and u_2v_2, we get that at most one of these edges is now dangerous. One can verify that we get $d(f', M) \leq d(f, M)-1$, while $s(f', M) \geq s(f, M)$; hence $d(f', M) - s(f', M) \leq d(f, M) + s(f, M) - 1 \leq \ell$, and the result follows by induction hypothesis. The other cases can be proved similarly.

4. Conclusion

In this paper, we prove that, given a graph G and a matching M of G, there exists a partial Grundy coloring of $G-M$ with at least $\partial\Gamma(G)/2$ colors. This shows that the analogue of Conjecture 3 to partial Grundy colorings holds, while we know that Conjecture 3 itself does not hold. It is interesting to note that this difference of behaviour is mainly due to the fact that $\Gamma(S_n \square H) \leq \Gamma(H)$, while the same is not true when dealing with partial Grundy colorings, as it is seen in Theorem 1.

Additionally, even knowing that Conjecture 3 does not hold for general graphs, it would be interesting to know whether there are graph classes in which it holds (for instance, it is known to hold for complete bipartite graphs), or whether it is possible to decide when the removal of a given matching cannot decrease the Grundy number by more than half.

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