On Tuza's conjecture in even co-chain graphs

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Abstract. *In 1981, Tuza conjectured that the cardinality of a minimum set of edges that intersects every triangle of a graph is at most twice the cardinality of a maximum set of edge-disjoint triangles. This conjecture has been proved for several important graph classes, as planar graphs, tripartite graphs, among others. However, it remains open on other important classes of graphs, as chordal graphs. Furthermore, it remains open for main subclasses of chordal graphs, as split graphs and interval graphs. In this paper, we show that Tuza's conjecture is valid for even co-chain graphs, a known subclass of interval graphs.*

1. Introduction

In this paper, all graphs considered are simple and the notation and terminology are standard. A *triangle hitting* of a graph G is a set of edges of G whose removal results in a triangle-free graph; and a *triangle packing* of G is a set of edge-disjoint triangles of G. We denote by $\tau(G)$ (resp. $\nu(G)$) the cardinality of a minimum triangle hitting (resp. maximum triangle packing) of G. Tuza posed the following conjecture.

Conjecture 1 ([Tuza 1981]) *For every graph G, we have $\tau(G) \leq 2\,\nu(G)$.*

This conjecture was verified for many classes of graphs [Tuza 1990, Cui and Wang 2009, Haxell 1999, Haxell et al. 2012] . However, Tuza's conjecture remains open for several important graph classes, as chordal graphs. For chordal graphs, Botler *et al.* showed that Tuza's conjecture is valid if G is K_8-free [Botler et al. 2021, Corollary 3.6]. However, the conjecture is still open for several important subclasses of chordal graphs, as split graphs and interval graphs.

In this direction, Bonamy *et al.* [Bonamy et al. 2021] verified this conjecture for threshold graphs, that is, graphs that are both split graphs and cographs. Another important subclass of chordal graphs are interval graphs. Several algorithmic and structural problems have been studied in co-chain graphs (see for instance [Boyacı et al. 2018, Kijima et al. 2012]), a known subclass of unit-interval graphs (see [Brandstädt et al. 1999, Theorem 9.1.2]). Bonamy *et al.* showed Conjecture 1 is valid for co-chain graphs with both sides of the same size divisible by 4. In this paper, we improve this result by showing that Tuza's conjecture is valid for every even co-chain graph.

2. Preliminaries

Given graph G and a vertex v in G, we denote by $N(v)$ the set of neighbors of v. If $X \subseteq V(G)$, then we denote by $G[X]$ the subgraph induced by X. A *co-bipartite* graph is a graph whose complement is bipartite. That is, if G is a co-bipartite graph, then there is a partition $\{K_\ell, K_m\}$ of $V(G)$ such that $G[K_\ell]$ and $G[K_m]$ are cliques. We denote a co-bipartite graph G with such partition as $G[K_\ell, K_m]$. A *co-chain* graph is a co-bipartite graph in which the neighborhoods of the vertices in each side can be linearly ordered with respect to inclusion. That is, if $G[K_\ell, K_m]$ is a co-chain graph, we can rename the vertices of K_ℓ as $c_1, c_2, \ldots, c_{|K_\ell|}$ and the vertices of K_m as $d_1, d_2, \ldots, d_{|K_m|}$ such that $N(c_{i+1}) \subseteq N(c_i)$ for $i = 1, 2, \ldots, |K_\ell| - 1$ and $N(d_i) \subseteq N(d_{i+1})$ for $i = 1, 2, \ldots, |K_m| - 1$.

A co-chain graph $G[K_\ell, K_m]$ is called *even* if $|K_\ell|$ and $|K_m|$ are both divisible by four. Bonamy *et al.* [Bonamy et al. 2021] showed that if $G[K_\ell, K_m]$ is even and $|K_\ell| = |K_m|$, then $\tau(G) \leq 2\nu(G)$. In this paper we extend this result by eliminating the restriction of equality.

Theorem 1 *If G is an even co-chain graph, then $\tau(G) \leq 2\nu(G)$.*

Given a graph G and two disjoint sets of vertices S and K, we let $p_G(S, K)$ be a maximum triangle packing in $G[K \cup S]$ among those that do not consider any edge in $G[S]$ and let $p_G(K)$ be a maximum triangle packing in $G[K]$. If the context is clear, we denote them by $p(S, K)$ and $p(K)$ respectively. We say that two sets of vertices in a graph are *complete to each other* if they are disjoint and every vertex in one set is adjacent to any vertex of the other set.

For our proof, we use two known results.

Proposition 1 (see [Bonamy et al. 2021, Corollary 5]) *Let K be a clique and S an independent set such that they are complete to each other in a graph G. Then $|p(S, K)| \geq \frac{|K|-1}{2} . \min\{|S|, |K|\}$. Moreover, if $|K|$ is even, then $|p(S, K)| \geq \frac{|K|}{2} . \min\{|S|, |K| - 1\}$.*

Proposition 2 ([Feder and Subi 2012, Lemma 6]) $|p(K_n)| = \frac{1}{3}(\binom{n}{2} - k)$ *where $k = 0$ if n MOD 6 $\in \{1, 3\}$, $k = 4$ if n MOD 6 =5, $k = \frac{n}{2}$ if n MOD 6 $\in \{0, 2\}$, and $k = \frac{n}{2} + 1$ if n MOD 6 =4.*

3. Proof of Theorem 1

For the rest of this section, we fix an even co-chain graph $G[K_\ell, K_m]$. Recall that K_ℓ and K_m have size divisible by 4. We abuse notation and let $\ell := |K_\ell|/2$, and $m := |K_m|/2$. As mentioned in the introduction, the case $\ell = m$ was already proved by [Bonamy et al. 2021]. Thus, we may assume that $\ell \neq m$.

As in [Bonamy et al. 2021], we let K_ℓ^{top}, K_ℓ^{bot} for the top and the bottom half of K_ℓ, respectively, and similarly K_m^{top} and K_m^{bot} for the top and the bottom half of K_m. Let $X_\ell \subseteq K_\ell$, $X_m \subseteq K_m$ be the sets defined as follows: $c \in X_\ell$ if $K_m^{bot} \subseteq N(c)$, and $d \in X_m$ if $K_\ell^{top} \subseteq N(d)$. We set $x_\ell = |X_\ell|$ and $x_m = |X_m|$ (Figure 1). As G is a co-chain graph, $x_\ell \geq \ell$ implies that the set $X_\ell \supseteq K_\ell^{top}$ is complete to K_m^{bot}. Consequently, $x_m \geq m$. Similarly, $x_m \geq m$ implies that $x_\ell \geq \ell$. Therefore, $x_\ell \geq \ell$ if and only if $x_m \geq m$. The rest of the proof is divided in two cases, if either $x_\ell \geq \ell$ or not.

3.1. The case $x_\ell \geq \ell$

We define a triangle packing P_1 as follows.

$$P_1 := p(X_\ell, K_m^{bot}) \cup p(X_m \setminus K_m^{bot}, K_\ell^{top}) \cup p(K_m^{bot}, K_m^{top}) \cup p(K_\ell^{top}, K_\ell^{bot}).$$

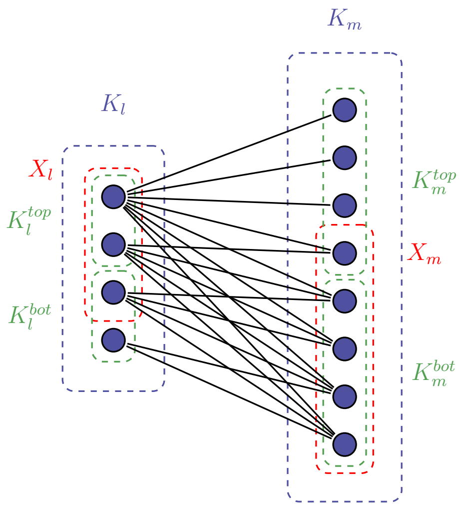

Figure 1. An even co-chain graph, with $\ell = 2, \mathbf{m} = 4, \mathbf{x}_\ell = 3$, and $\mathbf{x_m} = 5$. By simplicity, the edges inside $\mathbf{K}_\ell$ and $\mathbf{K_m}$ are not shown.

We choose a triangle hitting T_1 by taking all edges within K_ℓ^{top}, K_ℓ^{bot}, K_m^{top}, K_m^{bot} as well as all edges between K_ℓ^{top} and K_m^{bot}, and between K_ℓ^{bot} and K_m^{top}. We have

$$|T_1| \leq \ell m + (x_\ell - \ell)(x_m - m) + 2\binom{m}{2} + 2\binom{\ell}{2}.$$

Note that, by Proposition 1,

$$2|P_1| - |T_1| \geq m \cdot \min\{x_\ell, m-1\} + \ell \cdot \min\{x_m - m, l-1\} - \ell m - (x_\ell - \ell)(x_m - m). \quad (1)$$

For the rest of the proof in this case, we may assume, without loss of generality that $\ell < m$. Suppose for a moment that $\ell \leq x_\ell < m$. Then, by (1),

$$\begin{aligned} 2|P_1| - |T_1| &\geq m \cdot x_\ell + \ell \cdot \min\{x_m - m, \ell - 1\} - \ell m - (x_\ell - \ell)(x_m - m) \\ &\geq (2m - x_m)(x_\ell - \ell) \\ &\geq 0, \end{aligned}$$

and the proof follows. Hence, from now on, we may assume that $x_\ell \geq m > \ell$.

Note that, by (1), we have

$$\begin{aligned} 2|P_1| - |T_1| &\geq m \cdot (m-1) + \ell \cdot \min\{x_m - m, \ell - 1\} - \ell m - (x_\ell - \ell)(x_m - m) \\ &= m \cdot (m - \ell - 1) + \ell \cdot \min\{x_m - m, \ell - 1\} - (x_\ell - \ell)(x_m - m). \quad (2) \end{aligned}$$

Suppose for a moment that $x_m - m < \ell$. Then, by (2),

$$\begin{aligned} 2|P_1| - |T_1| &\geq m \cdot (m - \ell - 1) + \ell \cdot \min\{x_m - m, \ell - 1\} - (x_\ell - \ell)(x_m - m) \\ &= m \cdot (m - \ell - 1) + \ell \cdot (x_m - m) - (x_\ell - \ell)(x_m - m) \\ &= m \cdot (m - \ell - 1) + (2\ell - x_\ell) \cdot (x_m - m) \\ &\geq 0, \end{aligned}$$

and the proof follows. Thus, we may assume that $x_m - m \geq \ell$. The rest of the proof for this case can be analyzed with the same methodology.

Now, if $x_\ell < \ell$, we will select a clique with maximum size between $K_\ell^{top} \cup X_m$ and $K_m^{bot} \cup X_\ell$. If $K_m^{bot} \cup X_\ell$ has the maximum size, we define a triangle packing P as follows:

$$P := p(K_m^{bot} \cup X_\ell) \cup p(X_\ell \cup X_m, K_\ell^{top} \setminus X_\ell) \cup p(K_m^{bot}, K_m^{top}) \cup p(K_\ell^{top}, K_\ell^{bot}).$$

And also we choose a triangle hitting T obtained by taking all edges within K_ℓ^{top}, K_ℓ^{bot}, K_m^{top}, K_m^{bot} as well as edges between X_ℓ and K_m^{bot} and between X_m and K_ℓ^{top}. Then, in this case the idea is proof that $2|P| \geq |T|$.

4. Concluding Remarks

In this paper, we showed that Tuza's conjecture is valid for even co-chain graphs, that is, co-chain graphs where every side of the partition is divisible by four. We believe the technique can be adapted to extend this result by showing that Tuza's conjecture is valid for any co-chain graph. As mentioned by Bonamy *et al.*[Bonamy et al. 2021], a main motivation for this study is due the close relationship between co-chain graphs and unit interval graphs. Hence, our result is a step further to prove Tuza's conjecture in unit-interval graphs. Note that interval graphs are chordal graphs. We believe we are still far from proving Tuza's conjecture in chordal graphs. A step towards this goal would be to show Tuza's conjecture in split graphs, or subclasses of it.

References

Bonamy, M., Bożyk, Ł., Grzesik, A., Hatzel, M., Masařík, T., Novotná, J., and Okrasa, K. (2021). Tuza's conjecture for threshold graphs. In Extended Abstracts EuroComb 2021, pages 765–771. Springer.

Botler, F., Fernandes, C. G., and Gutiérrez, J. (2021). On tuza's conjecture for triangulations and graphs with small treewidth. Discrete Mathematics, 344(4):112281.

Boyacı, A., Ekim, T., and Shalom, M. (2018). The maximum cardinality cut problem in co-bipartite chain graphs. J. Comb. Optim., 35(1):250–265.

Brandstädt, A., Le, V. B., and Spinrad, J. P. (1999). Graph classes: a survey. SIAM.

Cui, Q. and Wang, J. (2009). Maximum bipartite subgraphs of cubic triangle-free planar graphs. Discrete Mathematics, 309(5):1091–1111.

Feder, T. and Subi, C. S. (2012). Packing edge-disjoint triangles in given graphs. In Electron. Colloquium Comput. Complex., volume 19, page 13.

Haxell, P. E. (1999). Packing and covering triangles in graphs. Discrete Mathematics, 195(1):251–254.

Haxell, P. E., Kostochka, A., and Thomassé, S. (2012). Packing and covering triangles in K_4-free planar graphs. Graphs and Combinatorics, 28(5):653–662.

Kijima, S., Otachi, Y., Saitoh, T., and Uno, T. (2012). Subgraph isomorphism in graph classes. Discrete Mathematics, 312(21):3164–3173.

Tuza, Z. (1981). Conjecture in: finite and infinite sets. In Proc. Colloq. Math. Soc. J. Bolyai (Eger, Hungary, 1981), volume 37, page 888.

Tuza, Z. (1990). A conjecture on triangles of graphs. Graphs and Combinatorics, 6(4):373–380.