Recognizing which Cographs are Set Graphs

Bruno Bandeira[1], Márcia R. Cerioli[2], Petrucio Viana[3]

[1]Programa de Engenharia de Sistemas e Computação – COPPE
Universidade Federal do Rio de Janeiro (UFRJ) – Rio de Janeiro – RJ – Brazil

[2]Instituto de Matemática e
Programa de Engenharia de Sistemas e Computação – COPPE
Universidade Federal do Rio de Janeiro (UFRJ) – Rio de Janeiro – RJ – Brazil

[3]Instituto de Matemática e Estatística
Universidade Federal Fluminense (UFF) – Niterói – RJ – Brazil

brunobm@cos.ufrj.br, marcia@cos.ufrj.br, petrucio_viana@id.uff.br

Abstract. *A* set graph *is a graph admitting an* extensional acyclic orientation. *The set graph recognition problem was first considered and proved to be NP-complete by A. Tomescu in 2012. In this work, we introduce two concepts that can be used for the efficient recognition of set graphs in some classes of graphs. We define* layered extensional acyclic orientations *and a graph parameter, called* set-deficiency*, that measures how far a graph is from being a set graph. Then, we describe how these concepts can be applied to recognize set graphs in the class of cographs in polynomial time.*

1. Introduction

A graph is *set graph* if it admits an extensional acyclic orientation (eao). An orientation of a graph is *extensional* if distinct vertices have distinct out-neighborhoods, and is *acyclic* if it has no cycles. A. Tomescu introduced the class of set graphs, and, together with E. Omodeo and A. Policriti, proved some results on the complexity of the *set graph recognition problem (EAO)*. Namely, they proved that EAO is NP-complete for bipartite graphs, can be solved in polynomial time for trees, unicyclic graphs, graphs admitting hamiltonian paths, claw-free graphs, complete multipartite graphs, block graphs, $(\text{apple}, K_{2,3}, \text{dart}, K_{1,1,3})$-free graphs, and it admits a linear-time recognition algorithm for classes of graphs with bounded *treewidth* (see [Tomescu 2012] and [Omodeo et al. 2017]). We contribute to this line of development by providing a polynomial-time algorithm for deciding whether a cograph is a set graph.

Cographs form a well-known class of graphs having many characterizations (cf. [Corneil et al. 1981]). In particular, a graph is a *cograph* if it has no P_4 as an induced subgraph.

To recognize set graphs in the class of cographs, we defined the concept of *layered extensional acyclic orientation* (leao) and a parameter called the *set-deficiency* of a graph. These concepts can be used to measure how far a graph is from being a set graph and so can be used to recognize set graphs efficiently in some classes of graphs. Finally, we show how to compute the set-deficiency of cographs in polynomial time.

2. Extensional Acyclic Orientations

All graphs are undirected and have no loops nor multiple edges. For every set X, we denote by $\mathcal{P}(X)$ the power set of X. For all $m, n \in \mathbb{N}$, $[m, n] = \{i \in \mathbb{N} : m \leq i \leq n\}$; and $[n] = [1, n] = \{1, \ldots, n\}$. For every set X and every $n \in \mathbb{N}$, $\binom{X}{n} = \{Y \in \mathcal{P}(X) : |Y| = n\}$.

Definition 1 Given a graph $G = (V, E)$, a set $D \subseteq V^2$ is an orientation of G if there is a bijection $dir_D : E \to D$ such that, for every edge $\{x, y\} \in E$, either $dir_D(\{x, y\}) = (x, y)$ or $dir_D(\{x, y\}) = (y, x)$. Given a vertex $v \in V(G)$ the out-neighborhood of v is defined by $N^+_D(v) := \{x \in V(G) : (v, x) \in D\}$.

For simplicity, we write xy instead of (x, y).

Definition 2 Given an orientation D of a graph G, a cycle in D is a sequence of vertices $x_1, x_2, \ldots, x_k$ such that $k > 1$, $x_1 = x_k$ and $x_i x_{i+1} \in D$ for every $i \in [k-1]$. We say that D is acyclic if D has no cycles.

Definition 3 An orientation D of a graph G is extensional if, for all $x, y \in V(G)$ with $x \neq y$, $N^+(x) \neq N^+(y)$. If, otherwise, two distinct vertices $x, y \in V(G)$ are such that $N^+_D(x) = N^+_D(y)$, we say that x and y collide.

Definition 4 [Tomescu 2012] A graph G is a set graph if G admits an extensional acyclic orientation (an eao).

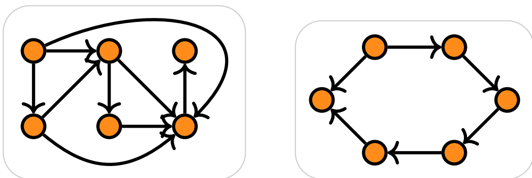

Figure 1. Extensional Acyclic Orientations

The orientations depicted in Figure 1 are examples of eaos. Hence, the underlying graphs are set graphs.

If D is an eao of a graph G, then D has at least one sink in each connected component because it is acyclic, but it has at most one sink because it is extensional. Thus, every set graph is connected.

An example of a connected graph that is not a set graph is the star $K_{1,n}$ for any $n \geq 3$. The out-neighborhoods of the leaves of the $K_{1,n}$ must be either the empty set or the singleton set formed by the center of the star, thus a star that is a set graph may have at most 2 leaves. More generally, a tree is a set graph iff it is a path; and a complete bipartite graph $K_{n,m}$ is a set graph iff $|n - m| \leq 1$ [Omodeo et al. 2017].

3. Layered Extensional Acyclic Orientations

The definition of layered extensional acyclic orientations (leaos) is obtained by relaxing the extensionality condition, allowing the graph to be separated into layers and only requiring that the given orientation be *extensional-by-layers* (i.e. out-neighborhoods only need to be distinct between pairs of vertices in the same layer), and *downwards* (i.e. directed edges cannot go from a lower to a higher layer). Leaos are central in our approach for solving EAO in the class of cographs.

Definition 5 Given a function $\ell : V(G) \to \mathbb{N}$ and an orientation D of a graph G, we say that the pair $\mathfrak{L} = (\ell, D)$ is: (a) extensional-by-layers when, for all $x, y \in V(G)$, if $x \neq y$ and $\ell(x) = \ell(y)$, then $N_D^+(x) \neq N_D^+(y)$; (b) downwards when, for all $x, y \in V(G)$, if $xy \in D$, then $\ell(x) \geq \ell(y)$; (c) a layered extensional acyclic orientation (leao) if D is acyclic and $\mathfrak{L}$ is extensional-by-layers and downwards.

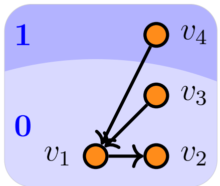

Figure 2. A 2-layers leao of $K_{1,3}$

Note that if every vertex is in the same layer in a leao $\mathfrak{L} = (\ell, D)$ of a graph G (i.e. $\ell(u) = \ell(v)$ for all $u, v \in V(G)$), then D is an eao. Hence, the fewer layers a leao has, the closer it is to being an eao.

Definition 6 *Given a leao $\mathfrak{L} = (\ell, D)$ of a graph G, the height of $\mathfrak{L}$, denoted $|\mathfrak{L}|$, is the highest layer of $\mathfrak{L}$, i.e. $|\mathfrak{L}| = \max\{\ell(v) : v \in V(G)\}$.*

Lemma 1 *A graph G is a set graph if and only if G admits a leao $\mathfrak{L}$ such that $|\mathfrak{L}| = 0$.*

Definition 7 *A leao $\mathfrak{L}$ of a graph G is minimum if $|\mathfrak{L}| \leq |\mathfrak{L}'|$ for every leao $\mathfrak{L}'$ of G.*

Figure 2 shows a minimum leao of $K_{1,3}$.

Definition 8 *The set-deficiency of a graph G, denoted $\mathcal{S}_\Delta(G)$, is the height of a minimum leao of G.*

If $\mathfrak{L} = (\ell, D)$ is a leao of a graph G with p connected components, then D has sinks in each connected component because it is acyclic, and each sink is in a different layer because $\mathfrak{L}$ is extensional-by-layers. Thus, if G has p connected components, then $\mathcal{S}_\Delta(G) \geq p - 1$.

4. Minimum Leaos of Cographs

The class of cographs has appeared naturally in many areas of mathematics and, in fact, was rediscovered independently by many authors (cf. [Corneil et al. 1981]).

Definition 9 *The class of cographs is the smallest class of graphs that contains the K_1 and is closed under disjoint unions and complementations.*

The definition of cographs can be rephrased into the following characterization.

Theorem 1 (cf. [Corneil et al. 1981]) *A graph $G = (V, E)$ is a cograph if and only if, for all $X \subseteq V$ with $|X| \geq 2$, the induced subgraph $G[X]$ is disconnected or has a disconnected complement.*

Inspired by the decomposition structure of a cograph, given by Theorem 1, we establish Lemmas 3 and 4 to determine the set-deficiency of a graph from the set-deficiency of some of its components.

The following Lemma 2, based on certain structural properties of cographs, guarantees that the assumptions of Lemma 3 are satisfied for graphs in this class.

Lemma 2 *Every cograph admits a minimum leao $\mathfrak{L} = (\ell, D)$ such that D has exactly one sink per connected component of G.*

Lemma 3 *Let G be a disconnected graph with connected components $G_0, \ldots, G_k$ such that, $\mathcal{S}_\Delta(G_0) \geq \ldots \geq \mathcal{S}_\Delta(G_k)$ and for each $i \in [0, k]$, G_i admits a minimum leao having exactly one sink. Then $\mathcal{S}_\Delta(G) = \max\{\mathcal{S}_\Delta(G_i) + i : i \in [0, k]\}$.*

The *join* of two graphs G_1 and G_2 is the graph $G_1 \wedge G_2$ obtained by taking the disjoint union of the two and connecting every vertex of G_1 to every vertex of G_2. Note that if a graph G has a disconnected complement, then it is the join of two induced subgraphs, H and $G \setminus X$, where $H = (X, E')$ is a connected component of the complement of G.

Lemma 4 *Let G_1 and G_2 be graphs such that $\mathcal{S}_\Delta(G_1) \geq \mathcal{S}_\Delta(G_2)$. Then $\mathcal{S}_\Delta(G_1 \wedge G_2) = \max\{\mathcal{S}_\Delta(G_1) - |V(G_2)|, 0\}$.*

Applying Lemmas 3 and 4 we obtain a simple method for computing the set-deficiency of any given cograph from the set-deficiencies of smaller cographs. Therefore, these lemmas can be used for recursively computing the set-deficiency of cographs in polynomial time.

Theorem 2 *EAO can be solved in polynomial time for cographs.*

5. Conclusion

We have extended our results on set-deficiencies further and were able to obtain a polynomial-time algorithm for recognizing set graphs in the class of P_4-sparse graphs. We envision that a broader study of leaos and set-deficiencies could lead to polynomial algorithms for solving EAO in larger graph classes. The layered extensional acyclic orientation has proved to be an interesting and rich concept, for which a few open problems have appeared.

These results will be presented in full detail in B. Bandeira's M.Sc. Thesis, along with a proof that determining if a split graph is a set graph is an NP-complete problem.

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil (CAPES) – Finance Code 001.

References

Corneil, D. G., Lerchs, H., and Burlingham, L. S. (1981). Complement reducible graphs. *Discrete Applied Mathematics*, 3(3):163–174.

Omodeo, E. G., Policriti, A., and Tomescu, A. I. (2017). *On sets and graphs: Perspectives on logic and combinatorics*. Springer.

Tomescu, A. I. (2012). *Sets as graphs*. PhD thesis, Università degli Studi di Udine.