Minimum Density of Identifying Codes of Hexagonal Grids with a Finite Number of Rows

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Abstract. *An identifying code (id code, for short) of a graph is a dominating set such that all vertices have a distinct closed neighbourhood within the code. We present a lower bound for the minimum density of id codes of infinite hexagonal grids with a finite number of rows. We also show that every id code that does not induce a trivial component has density at least* $3/7$*. Finally, we show that when such grids have two rows this minimum density is precisely* $9/20$*. The results on lower bounds are proved using the discharging method.*

1. Introduction

The concept of *id codes* was introduced by [Karpovsky et al. 1998] to identify a faulty processor in a multiprocessor system. The vertices of an id code correspond to special processors (the controllers) that are able to check themselves and their neighbours to identify a faulty processor. Problems on id codes have been studied on finite and infinite graphs, being of great interest both from theoretical as well as practical viewpoint. We refer the reader to an updated bibliography maintained by [Lobstein 2021] covering this topic and related ones. One of the problems that has raised much interest on this concept is that of finding id codes of minimum density, a problem known to be NP-hard [Charon et al. 2003]. On infinite graphs, these studies have considered grids with infinite or with a finite number of rows. Many results were obtained for square, triangular and king grids in both cases. Results on the minimum density of id codes of hexagonal grids are known only for grids with infinite number of rows. It has been shown that this parameter, denoted $d^*(\mathcal{G}_H)$, satisfies $5/12 \leq d^*(\mathcal{G}_H) \leq 3/7$ [Cukierman and Yu 2013, Cohen et al. 2000]. No such results have appeared in the literature for hexagonal grids H_k with fixed number k of rows. We show that every id code that does not induce a trivial component has density at least $3/7$. We also show that $d^*(H_k) \geq 2/5$, for all $k \geq 2$, and $d^*(H_2) = 9/20$. We use the discharging method to prove three lower bounds. We present the main idea behind this method and show how it is used to obtain the results in this context.

1.1. Hexagonal Grids, Identifying Codes and Minimum Density

A *hexagonal grid*, denoted $\mathcal{G}_H$, is an infinite graph with vertex set $V = \mathbb{Z} \times \mathbb{Z}$ and edge set $E = \{(u, v) : u = (i, j), u - v \in \{(\pm 1, 0), (0, (-1)^{i+j+1})\}\}$. A *hexagonal grid with k rows*, $k \geq 2$, denoted H_k, is a graph isomorphic to the subgraph of $\mathcal{G}_H$ induced by the vertex set $\mathbb{Z} \times \{1, \ldots, k\}$ (see Figure 1). Let G be a connected graph. If v is a vertex of

G, and r is a natural number, then $N_r(v)$ denotes the set of neighbors of v at distance at most r, and $N_r[v] = N_r(v) \cup \{v\}$. When $r = 1$, we omit it, and write $N(v)$ and $N[v]$. Given $C \subseteq V(G)$, let $C[v] = N[v] \cap C$. An *id code* of G is a set $C \subseteq V(G)$ such that: $C[v] \neq \emptyset$, for every vertex v of G, and $C[v] \neq C[w]$, for any pair of distinct vertices v, w of G. Not all graphs have an id code, those which have are called *identifiable*. If C is an id code, then we say that $C[v]$ is the *identifier* of v. If G is a finite or infinite graph, of bounded degree, the *density* $d(C, G)$ of an id code C of G is defined below. The *minimum density* of an id code of G, denoted by $d^*(G)$, is defined as $d^*(G) = \inf_{C \in \mathcal{C}}\{d(C, G)\}$, where $\mathcal{C}$ is the set of all id codes of G.

$$d(C, G) = \limsup_{r \to \infty} \frac{|C \cap N_r[v_0]|}{|N_r[v_0]|}, \text{ where } v_0 \text{ is an arbitrary vertex in } G.$$

1.2. Discharging Method

The discharging method is a proof technique in graph theory that has been used in many different contexts, such as in graph coloring, decomposition, embedding, geometric and structural problems. For a guide on the use of the this method to prove results on coloring and other structural properties of graphs, see [Cranston and West 2017]. Roughly speaking, to prove results on a graph G, this method involves two phases: *charging* and *discharging*. In the charging phase, we assign charges (a rational number) to certain structures of G using a *charging rule*, which describes the value of the charge and the structures of G which will receive the charge. These structures may be vertices, edges, faces (if G is planar), etc. In the discharging phase, we re-assign the charges using the *discharging rules*, which describe the structures that will send and/or receive charge. The discharging must preserve the total charge of the charging phase. The discharging rules are designed to guarantee that, after this phase, some information on the charges of certain vertices/edges will help us prove some property of the graph. In many applications, the initial charges or the discharging rules take into consideration the degree of the vertices. Here, to prove lower bounds on the minimum density of id codes, we use the discharging method as in Lemma 2.1, given in the next section.

2. A Lower Bound for the Minimum Density of an Identifying Code of H_k

Lemma 2.1. *Let G be a graph (possibly infinite) with bounded degree Δ, a fixed integer. Let C be an id code of G. To prove that $d(C, G) \geq q$ for some q, one may use the discharging method as follows: in the charging phase, assign charge 1 to each vertex in C and charge 0 to the remaining vertices. Then, one shows discharging rules that guarantee that, after using them, each vertex v of G has final charge $\mathrm{chg}(v)$ at least q.*

In [Karpovsky et al. 1998], it was proved that every finite d-regular graph G satisfies $d^*(G) \geq 2/(d + 2)$. This was done using a double counting argument on the set of possible identifying codes. In the next theorem we show that the same bound holds for every graph (possibly infinite) with bounded degree Δ. Our proof is based on the approach used by [Cranston and Yu 2009] to prove the lower bound $2/5$ for $d^*(\mathcal{G}_H)$.

Theorem 2.1. *Let Δ be a positive integer. If G is an identifiable graph with bounded degree Δ, then $d^*(G) \geq 2/(\Delta + 2)$; in particular, $d^*(H_k) \geq 2/5$ for any $k \geq 2$.*

Proof. Let C be an id code of G, and let $q = 2/(\Delta+2)$. We apply the discharging method with charging rules as stated in Lemma 2.1, and with the following discharging rule:

(R) If $v \notin C$ and $|C[v]| = d$, then v receives q/d of charge from each vertex in $C[v]$.

We prove now that $\mathrm{chg}(v) \geq q$ for every vertex v in G. Clearly, if $v \notin C$, then $\mathrm{chg}(v) = q$; so assume that $v \in C$. If v has no neighbor in C, then for all $w \in N(v)$ we have $|C[w]| \geq 2$, otherwise $C[v] = C[w]$. Thus, v sends at most $q/2$ of charge to each vertex in $N(v)$. As a vertex in G has degree at most Δ, it follows that charge of v after discharging is at least $1 - \Delta(q/2) = q$. Suppose now that v has a neighbor in C. Then for at most one vertex, say w, that is a neighbor of v outside C, we have that $C[w] = \{v\}$; and for all the remaining neighbors x of v outside C, we have that $|C[x]| \geq 2$. Thus v sends at most q of charge to w and at most $q/2$ to the remaining neighbors x in $N(v) \setminus C$. Since the degree of v is at most Δ, it follows that $\mathrm{chg}(v) \geq 1 - q - (\Delta - 2)(q/2) = q$. As $\mathrm{chg}(v) \geq q$ for every vertex v in G, by Lemma 2.1 we have that $d(C, G) \geq q$. □

In what follows, we prove that many id codes C of H_k satisfies $d(C, H_k) \geq 3/7$.

Theorem 2.2. *Let C be an id code of H_k ($k \geq 2$) such that every vertex in C has a neighbor in C. Then $d(C, H_k) \geq 3/7$.*

Proof. We apply the discharging method with charging rules as stated in Lemma 2.1, taking $q = 3/7$, and considering the following discharging rules:

(R1) If $v \notin C$ and $|C[v]| = d$, then v receives $3/(7d)$ of charge from each vertex in $C[v]$.

(R2) If $c \in C$ and $|N(c) \cap C| \geq 2$, c sends charge $1/14$ for each neighbor in $N(c) \cap C$.

We prove that $\mathrm{chg}(v) \geq 3/7$ for every vertex v, and conclude the proof using Lemma 2.1. Clearly, $\mathrm{chg}(v) = 3/7$ if $v \notin C$. Consider now a vertex $c \in C$. By hypothesis, we have that c has at least one neighbor in C. If c has exactly one neighbor c' in C, then c' must have other neighbor in C. Thus, c sends at most $3/7$ of charge to some neighbor $w \notin C$ and at most $3/14$ to the remaining neighbor x in $N(c) \setminus C$, and receives $1/14$ from c'. Hence, $\mathrm{chg}(c) \geq 1 - 3/7 - 3/14 + 1/14 = 3/7$. If c has exactly two neighbors in C, then c sends at most $3/7$ of charge to some neighbor $w \notin C$ and at most $1/14$ to each one of the two neighbors in C. Thus, $\mathrm{chg}(c) \geq 1 - 3/7 - 2(1/14) = 3/7$. If c has exactly three neighbors in C, then c sends at most $1/14$ of charge to each of them. Hence, $\mathrm{chg}(c) \geq 1 - 3(1/14) = 11/14 > 3/7$, and this concludes our proof. □

3. Minimum Density of an identifying code of H_2

The tile, say T, depicted in Figure 1, from column 1 to 20, generates a periodic tiling of H_2. Let C_2 be the 18 black vertices in T. We leave to the reader verify that C_2 is indeed an id code of T with density $9/20$. Thus, $d^*(H_2) \leq 9/20$.

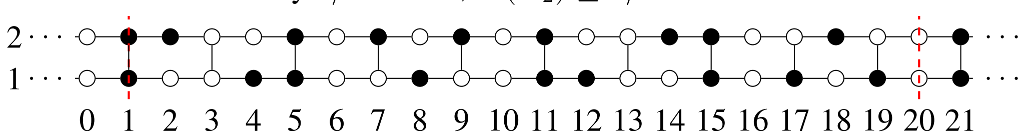

Figure 1. Pattern of an identifying code for H_2, whose density is $9/20$.

To show that $d^*(H_2) \geq 9/20$, we use the discharging method, but this time, we work with special 4-vertex sets. For an integer x, we say that a vertex of column x of

H_2 is *cubic* if it has degree 3 in H_2. Wlog, we consider that when x is odd then the vertices in columns x and $-x$ are cubic. For x odd, let Q_x be the quartet of vertices $\{(x,1),(x+1,1),(x,2),(x+1,2)\}$, see Figure 2. Note that $H_2[Q_x]$ is a P_4. Moreover, the vertices of H_2 can be partitioned into quartets Q_x. Given a quartet Q_x, we also refer to Q_{x-2} (resp. Q_{x+2}), its left (resp. right) quartet, as Q_x^L (resp. Q_x^R). We say that Q_x is *type* i (resp. *type* i^+) if $|Q_x \cap C| = i$ (resp. $|Q_x \cap C| \geq i$). A quartet Q_x is *satisfied* if the sum of their charges is at least $9/5$, otherwise it is *unsatisfied*. (If Q_x is satisfied, its charge can be distributed to its vertices, so that each vertex gets charge $9/20$).

Theorem 3.1. *The minimum density of any identifying code of* H_2 *is at least* $9/20$.

[*Sketch*] Take an id code C of H_2 and use the discharging method as stated in Lemma 2.1. Use the following discharging rules, and show that, after using them, each quartet Q_x is satisfied, and therefore, $d(C, H_2) \geq 9/20$.

(R1) If Q_x is type 0 and unsatisfied, then Q_x^L (resp. Q_x^R) sends $7/5$ (resp. $2/5$) to Q_x.
(R2) If Q_x is type 1 and unsatisfied, let y be the unique vertex in $Q_x \cap C$. If y is cubic, then Q_x^L (resp. Q_x^R) sends $1/5$ (resp. $3/5$) to Q_x; otherwise, Q_x^L (resp. Q_x^R) sends $3/5$ (resp. $1/5$) to Q_x.

(a) An example where Rule (R1) is applied.

(b) An example where Rule (R2) is applied.

Figure 2

4. Conclusion

It would be interesting to obtain, if possible, a better lower bound for $d^*(H_k)$, $k \geq 3$, and results on the upper bound.

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