Some results on irregular decomposition of graphs*

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Abstract. *A graph is locally irregular if any pair of adjacent vertices have distinct degrees. A locally irregular decomposition of a graph G is a decomposition of G into subgraphs that are locally irregular. We prove that any graph G can be decomposed into at most $2\Delta(G) - 1$ locally irregular graphs, improving on the previous upper bound of $3\Delta(G) - 2$. We also show some results on subcubic and non-decomposable graphs.*

1. Introduction

In this paper, we consider only connected and simple graphs. A graph is *locally irregular* if no pair of adjacent vertices have the same degree. A *k-locally irregular decomposition* (k-liec) of a graph G is a decomposition of G into at most k locally irregular subgraphs. Note that one can see a k-liec as an edge coloring $c\colon E(G) \to \{1, 2, \ldots, k\}$ in which the subgraphs induced by each color are locally irregular. The smallest k for which G admits a k-liec is its *irregular chromatic index*, denoted by $\chi'_{irr}(G)$. Figure 1 shows two examples of locally irregular decompositions.

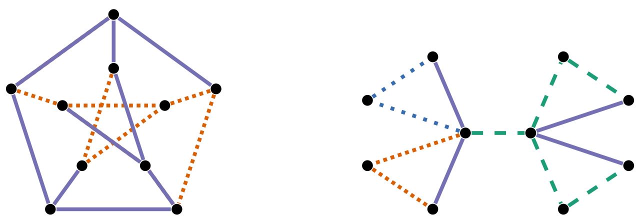

Figure 1. On the left, a 2-liec of the Petersen graph. On the right, a 4-liec of the *bow-tie graph*.

We observe that not all graphs admit a locally irregular decomposition, such as odd paths and odd cycles. We call graphs that admit such decompositions *decomposable*. Baudon, Bensmail, Przybyło, and Woźniak [Baudon et al. 2015] introduced the problem of determining $\chi'_{irr}(G)$ and characterized all non-decomposable graphs, which we present later. They also posed the following conjecture, which is related to the well-known 1-2-3 Conjecture [Karoński et al. 2004].

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Conjecture 1 ([Baudon et al. 2015]). *If G is a decomposable graph, then $\chi'_{irr}(G) \leq 3$.*

For information about the relation between the conjectures we refer the reader to [Baudon et al. 2019]. Conjecture 1 holds for all graphs with at most 9 vertices, cycles, complete graphs, complete bipartite graphs, trees, d-regular graphs with $d \geq 10^7$ [Baudon et al. 2015], bipartite cacti graphs [Bensmail 2020], cacti graphs with disjoint cycles [Lei et al. 2022], split graphs [Lintzmayer et al. 2021], and any graph G with $\delta(G) \geq 10^{10}$ [Przybyło 2015]. There are also partial results towards confirming the validity of Conjecture 1. For example, in [Lužar et al. 2018], it was proved that the irregular chromatic index is at most 7 for bipartite graphs and at most 4 for subcubic graphs, while in [Lei et al. 2022] it was shown that it is at most 4 for any cactus graph. If G is d-degenerate (e.g., planar graphs are 5-degenerate), then $\chi'_{irr}(G) \leq 3d + 1$ [Bensmail et al. 2020][1]. Finally, for general graphs G we have $\chi'_{irr}(G) \leq 220$ [Lužar et al. 2018], and if G is not regular, then we know that $\chi'_{irr}(G) \leq 3\Delta(G) - 2$ [Bensmail et al. 2020].

Sedlar and Škrekovski [Sedlar and Škrekovski 2021] noticed that the conjecture does not hold for a very small graph, the *bow-tie graph*, which has 10 vertices (see Figure 1). However, this is the only known counterexample. Therefore, one may ask if the bow-tie is the only counterexample for Conjecture 1. If the answer to that is negative, then the following questions remain:

(i) what is the minimum k such that $\chi'_{irr}(G) \leq k$ for any decomposable graph G that is not the bow-tie graph?
(ii) what are the graphs G for which $\chi'_{irr}(G) \leq 3$?

When trying to prove something on this problem by recursion, it is often the case one has to deal with subgraphs of a decomposable graph that are not decomposable. In Section 2, we discuss some useful structural results for dealing with such subgraphs and a result on subcubic decomposable graphs. In Section 3, we prove that in the case of general graphs G, we have $\chi'_{irr}(G) \leq 2\Delta(G) - 1$, which improves on the previous upper bound of $3\Delta(G) - 2$.

2. Non-decomposable graphs

In this section, we provide structural results on non-decomposable subcubic graphs. We start by describing the family $\mathcal{T}$ of non-decomposable graphs recursively as follows:

- $K_3 \in \mathcal{T}$;
- Let $G \in \mathcal{T}$ be a graph that contains a triangle with a vertex v of degree 2. Let H be an odd-length path with an endpoint identified with a vertex of a triangle, or let H be an even path. The graph obtained by identifying v with a vertex of H of degree 1 is also in $\mathcal{T}$.

See Figure 2 for examples.

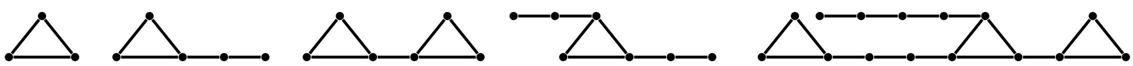

Figure 2. Examples of graphs in the family $\mathcal{T}$ of non-decomposable graphs.

[1]For planar graphs G, they proved that $\chi'_{irr}(G) \leq 15$.

Baudon, Bensmail, Przybyło, and Woźniak [Baudon et al. 2015] showed that a graph G is not decomposable if and only if G is in $\mathcal{T}$, or G is an odd-length cycle or an odd-length path. Note that all graphs that are not decomposable are subcubic. Also, graphs G in $\mathcal{T}$ have circumference 3 and every $v \in V(G)$ of degree 3 is contained in exactly one triangle.

The two following results guarantee the existence of 3-liecs in some graphs obtained by removing one edge of some graphs that are not decomposable. In what follows, we write P_n for a path with n vertices and we say P_n is an *even path* (*odd path*) if it contains an even (odd) number of edges. Furthermore, given a graph G and $v \in V(G)$, we denote the neighborhood of v in G by $N_G(v)$ and the degree of v by $d_G(v) := |N_G(v)|$.

Lemma 2. *Let G be a non-decomposable graph and let abc be a triangle of G such that $d_G(a) = 2$. There exists a 3-liec of $G - ab$ in which each color induces a forest of P_3.*

Proof. Since G contains a triangle, we know that $G \in \mathcal{T}$. The proof follows by induction on the number k of operations made to generate G by using the recursive definition of the family $\mathcal{T}$. If $k = 0$, then G is a triangle and $G - ab$ is a P_3, so the result clearly holds.

Now let G be built from identifying a degree-2 vertex v of $H \in \mathcal{T}$ with a degree-1 vertex z of a graph J which either is an odd path with an endpoint identified with a triangle or is an even path. Note that J is a 2-locally irregular decomposable graph, from where we conclude that the subgraph of J induced by each color is a forest of P_3. Also note that v is contained in a triangle of H.

If the triangle abc is contained in H, then, by induction, we know that $H - ab$ admits a 3-liec where each color induces a forest of P_3. Clearly, $v \in V(H - ab)$ "sees" at most two colors, which means we can color J with two colors, using in vz a color that is not seen by v. This yields the desired 3-liec of G.

If abc is not in H, then J must be an odd path with the triangle abc in one endpoint. Now let $H' = H - vy$, where y is a vertex that belongs to the same triangle as v. By induction, H' admits a 3-liec using only P_3. This means that there is one color not seen by v and y, which we can use to color vy and zz', where z' is the neighbor of z in J. If $J - ab$ is just an odd path, then $J - ab - zz'$ is an even path which admits a 2-liec using only P_3. Otherwise, $J - ab$ is an odd path with the degree-2 vertex of a P_3 identified in one endpoint. In this case, $J - ab - zz'$ also admits a 2-liec using only P_3. This yields the desired 3-liec of G. □

Lemma 3. *Let G be a non-decomposable graph. For any a and b such that $d_G(a) = 1$ and $b \in N_G(a)$, there exists a 3-liec of $G - ab$ in which each color induces a forest of P_3.*

Given a graph G, a vertex $u \in V(G)$, and $v \notin V(G)$, *hanging v in u* means building a new graph G' with $V(G') = V(G) \cup \{v\}$ and $E(G') = E(G) \cup \{uv\}$. The following results describe what happens when we add one edge to a graph that is not decomposable.

Lemma 4. *Let G be a non-decomposable graph, $u \in V(G)$, $v \notin V(G)$, and let G' be a graph obtained by hanging v in u. If $d_G(u) = 2$ and u is not in a triangle of G, then G' admits a 3-liec where u is monochromatic and G' also admits a 3-liec in which each color induces a forest of P_3.*

Lemma 5. *Let G be a non-decomposable graph, $u \in V(G)$, $v \notin V(G)$, and let G' be a graph obtained by hanging v in u. If u is in a triangle of G or $d_G(u) = 1$, then G' admits a 3-liec in which each color induces a forest of P_3.*

Lemma 6. *Let G be a non-decomposable graph, $u \in V(G)$, $v \notin V(G)$, and let G' be a graph obtained by hanging v in u. If $d_G(u) = 3$, then G' admits a 3-liec in which each color induces a forest of P_3.*

Putting Lemmas 4, 5, and 6 together, we obtain the following corollary.

Corollary 7. *Let G be a non-decomposable graph, $u \in V(G)$, $v \notin V(G)$, and let G' be a graph obtained by hanging v in u. Then $\chi'_{irr}(G') \leq 3$. Furthermore, G' admits a 3-liec in which each color induces a forest of P_3.*

Lemma 8. *Let G be a non-decomposable graph, $u, v \in V(G)$ with $uv \notin E(G)$, and let $G' = G + uv$. Then G' admits a 4-liec in which each color induces a forest of P_3 and one of the four colors appears only at u.*

Proof. Let $z \notin V(G)$ and let G'' be obtained from G by hanging z in u. Consider a 3-liec c of G'' according to Corollary 7. Let $w \in N_{G''}(u)$ be such that $c(uw) = c(uz)$. In G', give a fourth color (not used by c) to the edges uw and uv. Then, for the rest of the edges give the same color as in G''. This yields a 4-liec of G' with the desired properties. □

3. General upper bound for the irregular chromatic index

In this section, we prove the following result that gives an upper bound on $\chi'_{irr}(G)$ for any decomposable graph G based on $\Delta(G)$ and improves on the previous best-known result.

Theorem 9. *If G is a decomposable graph, then $\chi'_{irr}(G) \leq 2\Delta(G) - 1$.*

Outline of the proof. The proof follows by induction on the number of edges of G. If $|E(G)| = 2$, then G is a P_3 and clearly $\chi'_{irr}(G) = 1 \leq 2 \cdot 2 - 1 = 3$.

Assume $|E(G)| \geq 3$. If G is an even cycle or path, then $\chi'_{irr}(G) \leq 3 \leq 2 \cdot 2 - 1 = 3$. If G is subcubic with $\Delta(G) = 3$, then we know that $\chi'_{irr}(G) \leq 4 \leq 2 \cdot 3 - 1 = 5$ [Lužar et al. 2018]. Therefore, $\Delta(G) \geq 4$. Let $v \in V(G)$ with $d_G(v) = \Delta(G)$. Let $H = G - v$ and let $H_1, \ldots, H_t$ be the connected components of H. Consider that $H_1, \ldots, H_k$ are the components that induce non-decomposable graphs and let $H_{k+1}, \ldots, H_t$ be the decomposable ones.

For each non-decomposable component H_j, with $1 \leq j \leq k$, let $H'_j = H_j + vu_j$, where $u_j \in V(H_j) \cap N_G(v)$. By Corollary 7, H'_j admits a 3-liec where each color induces a forest of P_3. Each decomposable component H_j, with $k < j \leq t$, admits a locally irregular decomposition with at most $2\Delta(H_j) - 1 \leq 2\Delta(G) - 1$ colors by induction.

Now there are $\Delta(G) - k$ edges incident to v which are not colored. We show that, if $\Delta(G) - k$ is even, then we form $(\Delta(G) - k)/2$ pairs of such edges, and if $\Delta(G) - k$ is odd, then we form one triplet and $(\Delta - k - 3)/2$ pairs, such that each group can be colored without using new colors.

The rest of the proof follows by analyzing some cases according to the value of k. □

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