On the conformable colorings of k-regular graphs*

Luerbio Faria [1], Mauro Nigro [1], Diana Sasaki [1]

[1]Universidade do Estado do Rio de Janeiro (UERJ)
Rio de Janeiro – RJ – Brazil

{luerbio, diana.sasaki}@ime.uerj.br, mauro.nigro@pos.ime.uerj.br

Abstract. *In 1988, Chetwynd and Hilton defined conformable vertex colorings when trying to characterize the vertex colorings induced by a* $(\Delta + 1)$*-total coloring. Anticonformable colorings were used to characterize the subcubic conformable graphs. A graph* G *is anticonformable if it has a* $(\Delta + 1)$*-vertex coloring such that the number of color classes (including empty color classes) with the same parity as* $|V|$ *is at most* $def(G) = \sum_{v \in V} (\Delta - d_G(v))$*. The only connected subcubic not anticonformable graph is the triangular prism graph* L_3*. In this paper, we prove that if* k *is even, then every* k*-regular graph is not anticonformable; and if* $k \geq 3$ *is odd, then there is a not anticonformable graph* H_k*, where* $H_3 = L_3$*.*

1. Introduction

A *proper* k*-vertex coloring* of G is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. The colors are denoted by natural numbers and a class of color $i \in \mathbb{N}$ by $\mathcal{C}_i$. A *proper* k*-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. In this paper, all colorings are proper, and thus we omit the proper term. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta + 1$. The Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta + 2$ [Behzad 1965, Vizing 1964]. If the TCC holds in general, then the graphs can be partitioned into 2 collections: graphs with $\chi''(G) = \Delta + 1$, called *Type 1*, and graphs with $\chi''(G) = \Delta + 2$, called *Type 2*.

The *deficiency* of G is $def(G) = \sum_{v \in V} (\Delta - d(v))$, where $d(v)$ is the degree of a vertex v in G. A graph G is *conformable* if G has a $(\Delta + 1)$-vertex coloring φ in which the number of color classes (including empty color classes) whose parity differs from that of $|V(G)|$ is at most $def(G)$. In this case, we say that φ is a *conformable coloring*. If G is not conformable, then G is said *non-conformable*. Note that if G is a regular graph, then $def(G) = 0$ and φ is called conformable if and only if each color class has the same parity as $|V(G)|$. In Figure 1 we depict the 3 possible cases for graphs according to being *Type 1* or *Type 2* and their conformable classification. The *disjoint union* of graphs G and H is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, where $V(G) \cap V(H) = \emptyset$. The disjoint union of $\lambda \geq 2$ copies of a graph G is denoted by λG.

[Chetwynd and Hilton 1988] studied total coloring and observed that every $(\Delta + 1)$-total coloring of a graph induces a special $(\Delta + 1)$-vertex coloring. They defined

*This work was carried out with the partial support of the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Funding Code 001, of CNPq (306218/2022-4, 406036/2021-7, 313797/2020-0) and of FAPERJ (E26/201.360/2021 JCNE, E-26/010.002674/2019 ARC, E-26/200.519/2023 CNE)

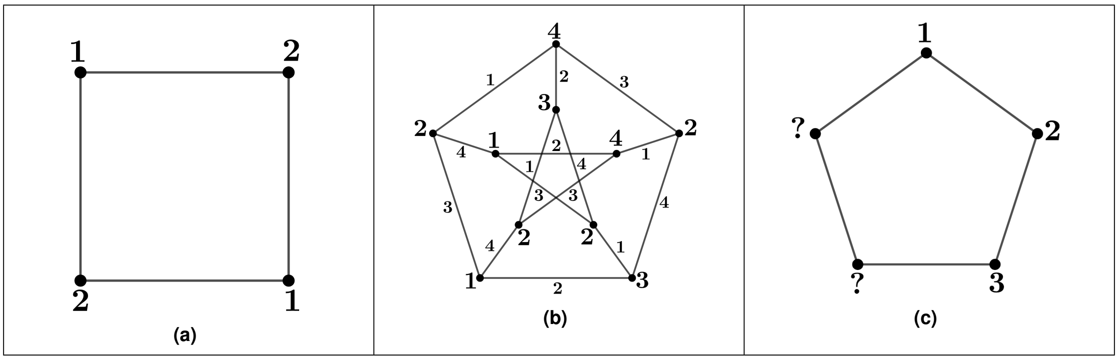

Figure 1. In (1a), the cycle graph C_4 which is conformable, but is *Type 2*. In (1b), the Petersen graph which is *Type 1* and, consequently, conformable. In (1c), the cycle graph C_5 which is non-conformable and, consequently, is *Type 2*. Note that vertices with $?$ cannot be assigned 1, 2, or 3, since these color classes must be odd.

this as a conformable coloring. Using this definition, they introduced the conformable graph class, consisting of graphs that admit conformable colorings. Therefore, *Type 1* graphs are contained within the conformable graph class. Several studies involving the CONFORMABLE VERTEX COLORING problem have been considered.

CONFORMABLE VERTEX COLORING
Instance: A graph $G = (V, E)$.
Question: Is there a $(\Delta + 1)$-vertex coloring φ in which the number of color classes (including empty color classes) with different parity of $|V|$ is at most $def(G)$ (i.e., is G conformable)?

[Hamilton et al. 1999] gave necessary conditions for a graph to be non-conformable. [Hilton and Hind 2002] proved that for non-conformable graphs $\Delta - 2$ is an upper bound for the deficiency of G. [Campos and de Mello 2007] posed a conjecture with a characterization of *Type 1* power of cycle graphs. [Nigro et al. 2021] proved that there are infinite families of circulant graphs *Type 1* and proved that there is a *Type 2* circulant graph that is conformable. [Zorzi et al. 2022] proved that the characterization of Campos and de Mello holds to the conformable power of cycle graphs. [Nigro et al. 2022b] classified the connected conformable subcubic graphs. A graph G is said *subcubic* if $\Delta(G) = 3$. Recently, [Nigro et al. 2022a] introduced the concept of anticonformable coloring in order to establish the general classification of conformable subcubic graphs.

Theorem 1 ([Nigro et al. 2022a])**.** *Let G be a subcubic graph. Then G is non-conformable and not Type 1 if and only if*

1. $G = \lambda K_4$, *where λ is an odd positive integer.*
2. $G = \lambda K_4 \cup K_{3,3}$, *where λ is an even positive integer.*
3. $G = \lambda K_4 \cup L_3$, *where λ is an odd positive integer.*

Notice that there are graphs that are not *Type 1* and conformable (Figure 1a). In this paper, we use anticonformable coloring as a tool in the process of classifying conformable graphs. We prove that if k is even, then every k-regular graph is not anticonformable; and that if k is odd, then there is a not anticonformable graph H_k. We prove that $\lambda K_{2q+2} \cup H_{2q+1}$ is non-conformable, where λ is an odd positive integer and $q \geq 1$.

2. Main result

A graph G is *anticonformable* if it has a $(\Delta(G)+1)$-vertex coloring φ in which the number of color classes (including empty color classes) with the same parity as $|V(G)|$ is at most $def(G)$. In this case, we say that φ is an *anticonformable coloring*. Note that if G is a regular graph, then φ is called anticonformable if the parity of each color class differs from that of $|V(G)|$. Figures 2a and 2b present examples of subcubic graphs with the corresponding anticonformable classification. Figure 2c presents the cycle graph C_5, which is not anticonformable.

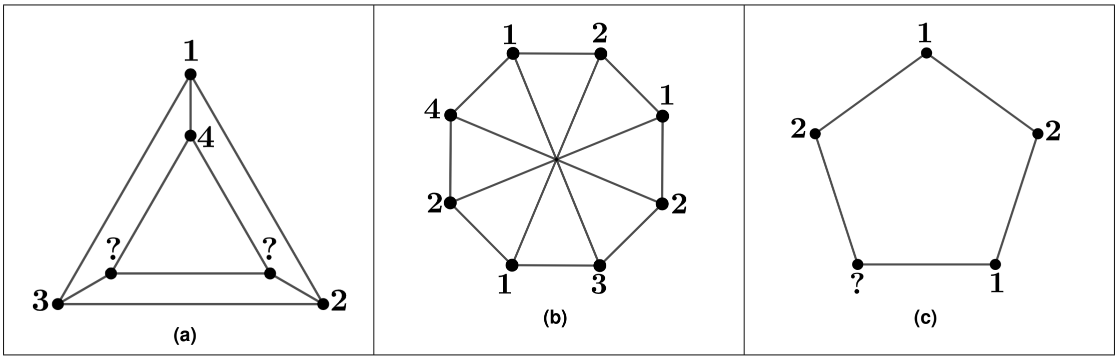

Figure 2. In (2a), the triangular prism graph L_3 is not anticonformable. Note that vertices with $?$ cannot be assigned 1, 2, or 3, since these color classes must be odd. In (2b), the Möbius Ladder M_8 with 8 vertices is anticonformable. In (2c), the cycle graph C_5 is not anticonformable. Note that the vertex with $?$ cannot be assigned 1, 2, or 3, since these color classes must be even.

Theorem 2. *If G is k-regular with k even, then G is not anticonformable.*

Proof. Let G be a k-regular graph with k even. Suppose by contradiction that there is an anticonformable coloring φ to G. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{k+1}$ be the color classes of φ. By definition of anticonformable and since $def(G) = 0$, the parity of each color class differs from that of $|V(G)|$. We consider the two cases,

1. Suppose that $|V(G)|$ is even. Since $|\mathcal{C}_i|$ is odd for $1 \le i \le k+1$, then $|V(G)| = |\mathcal{C}_1| + |\mathcal{C}_2| + \cdots + |\mathcal{C}_{k+1}|$ is odd, resulting in a contradiction.
2. Suppose that $|V(G)|$ is odd. Since $|\mathcal{C}_i|$ is even, for $1 \le i \le k+1$, then $|V(G)| = |\mathcal{C}_1| + |\mathcal{C}_2| + \cdots + |\mathcal{C}_{k+1}|$ is even, resulting in a contradiction as well.

Therefore, G is not anticonformable. $\square$

Let q be a positive integer. The k-regular graph H_k, for $k = 2q+1$ is defined as $V(H_k) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_{q+1}\}$ and $V = \{v_0, v_1, \ldots, v_{q+1}\}$ are cliques of H_k and $E(H_k) = \{u_iu_j, v_iv_j \mid i \neq j \text{ and } i,j \in \{0,\ldots,q+1\}\} \cup \{u_iv_i, u_iv_{i+1}, \ldots, u_iv_{i+(q-1)} \mid i \in \{0,\ldots,q+1\}\}$, where the index p of v_p is taken $p \mod (q+2)$. Note that $|V(H_k)| = 2q+4$. The graph H_3 is isomorphic to the triangular prism graph. In Figure 3 we present a drawing for H_5 and H_7.

Theorem 3. *Let q be a positive integer with $k = 2q+1$. Then the k-regular graph H_k is not anticonformable.*

Proof. Suppose, by contradiction, that H_k is anticonformable. Let φ be an anticonformable coloring of H_k. Let $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_{2q+2}$ be the color classes of φ. From the definition of anticonformable coloring, as $def(H_k) = 0$ and $|V(H_k)|$ is even, each color class $\mathcal{C}_i$ is odd. Hence, every color class is non-empty. Since U and V form a partition into cliques for $V(H_k)$, each color class $\mathcal{C}_i$ is singleton. Hence, $|\mathcal{C}_1| + |\mathcal{C}_2| + \cdots + |\mathcal{C}_{2q+2}| = 2q+2 \neq 2q+4 = |V(H_k)|$, a contradiction. Therefore, H_k is not anticonformable. $\square$

Corollary 1. *Let λ and q be positive integers with $k = 2q+1$. If λ is odd, then the graph $\lambda K_{2q+2} \cup H_k$ is non-conformable.*

Proof. We remark that $\Delta(K_{2q+2}) = \Delta(H_k) = 2q+1$. Since in any $(2q+2)$-vertex coloring to K_{2q+2} has each color class singleton, λK_{2q+2} has each color class with size λ. If λ is odd, then λK_{2q+2} is non-conformable. From Theorem 3, H_k has no anticonformable coloring, i.e., any $(2q+2)$-vertex coloring of H_k has at least one even color class. Hence, any $(2q+2)$-vertex coloring to $\lambda K_{2q+2} \cup H_k$ has at least one odd color class. Since $|V(\lambda K_{2q+2} \cup H_k)|$ is even and $def(\lambda K_{2q+2} \cup H_k) = 0$, $\lambda K_{2q+2} \cup H_k$ is non-conformable. $\square$

Corollary 2. *Let λ and q be positive integers with $k = 2q+1$. If λ is odd, then the graph $\lambda K_{2q+2} \cup H_k$ is not Type 1 and non-conformable.*

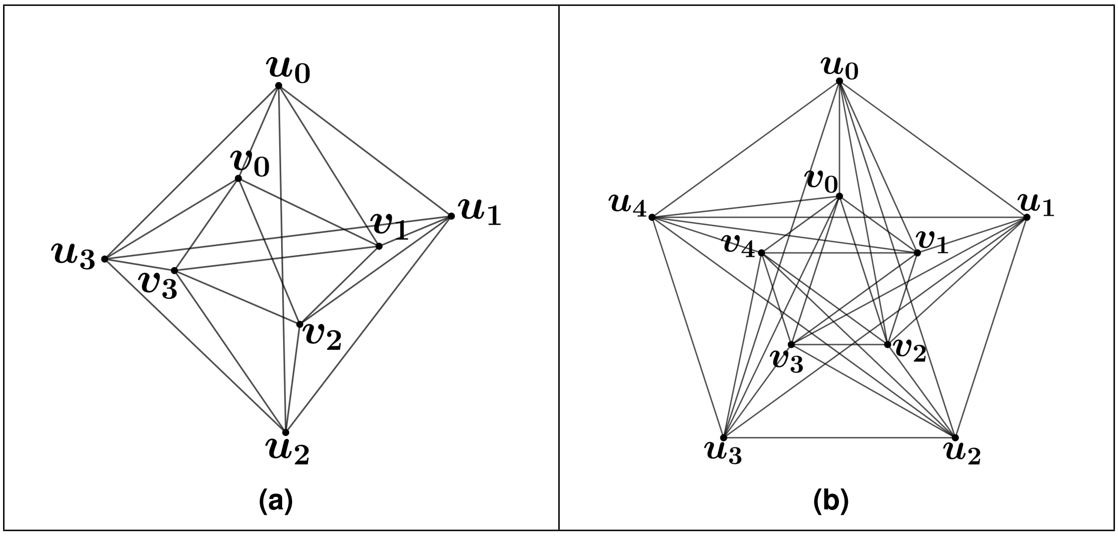

Figure 3. In (3a), the graph $H_5 = H_{2\cdot2+1}$, where $V(H_5) = \{u_0, u_1, u_2, u_3\} \cup \{v_0, v_1, v_2, v_3\}$ and $|V(H_5)| = 2\cdot 2+4$. In (3b), the graph $H_7 = H_{2\cdot3+1}$, where $V(H_7) = \{u_0, u_1, u_2, u_3, u_4\} \cup \{v_0, v_1, v_2, v_3, v_4\}$ and $|V(H_7)| = 2\cdot 3+4$.

Futher work

In this work, we present the classification of anticonformable k-regular graphs with k even and we prove that there is a not anticonformable k-regular graph with k odd. Finally, we show an application of the anticonformable definition in order to determine a family of not *Type 1* graphs that are non-conformable. As a future work, we aim to characterize, for each integer $k \geq 4$, the family of graphs with maximum degree $\Delta = k$ that are not anticonformable. Finally, we intend to use this result in order to establish the classification of not *Type 1* graphs with maximum degree $\Delta = k$ that are non-conformable.

References

Behzad, M. (1965). *Graphs and their chromatic numbers*. PhD thesis, Michigan State University.

Campos, C. N. and de Mello, C. P. (2007). A result on the total colouring of powers of cycles. *Discret. Appl. Math.*, 155:585–597.

Chetwynd, A. G. and Hilton, A. J. W. (1988). Some refinements of the total chromatic number conjecture. *Congr. Numer.*, pages 195–216.

Hamilton, G. M., Hilton, A. J. W., and Hind, H. R. F. (1999). Totally critical even order graphs. *J. Comb. Theory Ser. B.*, 76(2):262–279.

Hilton, A. J. W. and Hind, H. R. (2002). Non-conformable subgraphs of non-conformable graphs. *Discrete Math.*, 256(1):203–224.

Nigro, M., Adauto, M. N., and Sasaki, D. (2021). On total coloring of 4-regular circulant graphs. *Procedia Computer Science*, 195:315–324.

Nigro, M., Faria, L., and Sasaki, D. (2022a). A conformabilidade dos grafos subcúbicos. In *Anais do LIV Simpósio Brasileiro de Pesquisa Operacional*. Galoá.

Nigro, M., Faria, L., and Sasaki, D. (2022b). A conformabilidade dos grafos subcúbicos conexos. In *Anais do VII Encontro de Teoria da Computação*, pages 49–52.

Vizing, V. (1964). On an estimate of the chromatic class of a p-graph. *Metody Diskret. Analiz.*, 3:25–30.

Zorzi, A., Figueiredo, C., Machado, R., Zatesko, L., and Souza, U. (2022). Compositions, decompositions, and conformability for total coloring on power of cycle graphs. *Discret. Appl. Math.*, 323:349–363.