On the conformable colorings of $k$-regular graphs

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Abstract. In 1988, Chetwynd and Hilton defined conformable vertex colorings when trying to characterize the vertex colorings induced by a $(\Delta + 1)$-total coloring. Anticonformable colorings were used to characterize the subcubic conformable graphs. A graph $G$ is anticonformable if it has a $(\Delta + 1)$-vertex coloring such that the number of color classes (including empty color classes) with the same parity as $|V|$ is at most $\text{def}(G) = \sum_{v \in V} (\Delta - d_G(v))$. The only connected subcubic not anticonformable graph is the triangular prism graph $L_3$. In this paper, we prove that if $k$ is even, then every $k$-regular graph is not anticonformable; and if $k \geq 3$ is odd, then there is a not anticonformable graph $H_k$, where $H_3 = L_3$.

1. Introduction

A proper $k$-vertex coloring of $G$ is an assignment of $k$ colors to the vertices of $G$ so that adjacent vertices have different colors. The colors are denoted by natural numbers and a class of color $i \in \mathbb{N}$ by $C_i$. A proper $k$-total coloring of $G$ is an assignment of $k$ colors to the vertices and edges of $G$ so that adjacent or incident elements have different colors. In this paper, all colorings are proper, and thus we omit the proper term. The total chromatic number of $G$, denoted by $\chi''(G)$, is the smallest $k$ for which $G$ has a $k$-total coloring. Clearly, $\chi''(G) \geq \Delta + 1$. The Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta + 2$ [Behzad 1965, Vizing 1964]. If the TCC holds in general, then the graphs can be partitioned into 2 collections: graphs with $\chi''(G) = \Delta + 1$, called Type 1, and graphs with $\chi''(G) = \Delta + 2$, called Type 2.

The deficiency of $G$ is $\text{def}(G) = \sum_{v \in V} (\Delta - d(v))$, where $d(v)$ is the degree of a vertex $v$ in $G$. A graph $G$ is conformable if $G$ has a $(\Delta + 1)$-vertex coloring $\varphi$ in which the number of color classes (including empty color classes) whose parity differs from that of $|V(G)|$ is at most $\text{def}(G)$. In this case, we say that $\varphi$ is a conformable coloring. If $G$ is not conformable, then $G$ is said non-conformable. Note that if $G$ is a regular graph, then $\text{def}(G) = 0$ and $\varphi$ is called conformable if and only if each color class has the same parity as $|V(G)|$. In Figure 1 we depict the 3 possible cases for graphs according to being Type 1 or Type 2 and their conformable classification. The disjoint union of graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, where $V(G) \cap V(H) = \emptyset$. The disjoint union of $\lambda \geq 2$ copies of a graph $G$ is denoted by $\lambda G$.

[Chetwynd and Hilton 1988] studied total coloring and observed that every $(\Delta + 1)$-total coloring of a graph induces a special $(\Delta + 1)$-vertex coloring. They defined
Figure 1. In (1a), the cycle graph $C_4$ which is conformable, but is Type 2. In (1b), the Petersen graph which is Type 1 and, consequently, conformable. In (1c), the cycle graph $C_5$ which is non-conformable and, consequently, is Type 2. Note that vertices with ? cannot be assigned 1, 2, or 3, since these color classes must be odd.

CONFORMABLE VERTEX COLORING

Instance: A graph $G = (V, E)$.

Question: Is there a $(\Delta + 1)$-vertex coloring $\phi$ in which the number of color classes (including empty color classes) with different parity of $|V|$ is at most $\text{def}(G)$ (i.e., is $G$ conformable)?

[Hamilton et al. 1999] gave necessary conditions for a graph to be non-conformable. [Hilton and Hind 2002] proved that for non-conformable graphs $\Delta - 2$ is an upper bound for the deficiency of $G$. [Campos and de Mello 2007] posed a conjecture with a characterization of Type 1 power of cycle graphs. [Nigro et al. 2021] proved that there are infinite families of circulant graphs Type 1 and proved that there is a Type 2 circulant graph that is conformable. [Zorzi et al. 2022] proved that the characterization of Campos and de Mello holds to the conformable power of cycle graphs. [Nigro et al. 2022b] classified the connected conformable subcubic graphs. A graph $G$ is said subcubic if $\Delta(G) = 3$. Recently, [Nigro et al. 2022a] introduced the concept of anticonformable coloring in order to establish the general classification of conformable subcubic graphs.

Theorem 1 ([Nigro et al. 2022a]). Let $G$ be a subcubic graph. Then $G$ is non-conformable and not Type 1 if and only if

1. $G = \lambda K_4$, where $\lambda$ is an odd positive integer.
2. $G = \lambda K_4 \cup K_{3,3}$, where $\lambda$ is an even positive integer.
3. $G = \lambda K_4 \cup L_3$, where $\lambda$ is an odd positive integer.

Notice that there are graphs that are not Type 1 and conformable (Figure 1a). In this paper, we use anticonformable coloring as a tool in the process of classifying conformable graphs. We prove that if $k$ is even, then every $k$-regular graph is not anticonformable; and that if $k$ is odd, then there is a not anticonformable graph $H_k$. We prove that $\lambda K_{2q+2} \cup H_{2q+1}$ is non-conformable, where $\lambda$ is an odd positive integer and $q \geq 1$. 
2. Main result

A graph $G$ is anticonformable if it has a $(\Delta(G) + 1)$-vertex coloring $\varphi$ in which the number of color classes (including empty color classes) with the same parity as $|V(G)|$ is at most $\text{def}(G)$. In this case, we say that $\varphi$ is an anticonformable coloring. Note that if $G$ is a regular graph, then $\varphi$ is called anticonformable if the parity of each color class differs from that of $|V(G)|$. Figures 2a and 2b present examples of subcubic graphs with the corresponding anticonformable classification. Figure 2c presents the cycle graph $C_5$, which is not anticonformable.

![Figure 2](image)

**Figure 2.** In (2a), the triangular prism graph $L_3$ is not anticonformable. Note that vertices with ? cannot be assigned 1, 2, or 3, since these color classes must be odd. In (2b), the Möbius Ladder $M_8$ with 8 vertices is anticonformable. In (2c), the cycle graph $C_5$ is not anticonformable. Note that the vertex with ? cannot be assigned 1, 2, or 3, since these color classes must be even.

**Theorem 2.** If $G$ is $k$-regular with $k$ even, then $G$ is not anticonformable.

**Proof.** Let $G$ be a $k$-regular graph with $k$ even. Suppose by contradiction that there is an anticonformable coloring $\varphi$ to $G$. Let $C_1,C_2,\ldots,C_{k+1}$ be the color classes of $\varphi$. By definition of anticonformable and since $\text{def}(G) = 0$, the parity of each color class differs from that of $|V(G)|$. We consider the two cases,

1. Suppose that $|V(G)|$ is even. Since $|C_i|$ is odd for $1 \leq i \leq k+1$, then $|V(G)| = |C_1| + |C_2| + \cdots + |C_{k+1}|$ is odd, resulting in a contradiction.

2. Suppose that $|V(G)|$ is odd. Since $|C_i|$ is even, for $1 \leq i \leq k+1$, then $|V(G)| = |C_1| + |C_2| + \cdots + |C_{k+1}|$ is even, resulting in a contradiction as well.

Therefore, $G$ is not anticonformable.

Let $q$ be a positive integer. The $k$-regular graph $H_k$, for $k = 2q + 1$ is defined as $V(H_k) = U \cup V$, where $U = \{u_0, u_1, \ldots, u_{q+1}\}$ and $V = \{v_0, v_1, \ldots, v_{2q+1}\}$ are cliques of $H_k$ and $E(H_k) = \{u_i u_j, v_i v_j \mid i \neq j \text{ and } i, j \in \{0, \ldots, q+1\}\} \cup \{u_i v_i, u_i v_{i+1}, \ldots, u_i v_{i+(q-1)} \mid i \in \{0, \ldots, q+1\}\}$, where the index $p$ of $v_p$ is taken $p \mod (q + 2)$. Note that $|V(H_k)| = 2q + 4$. The graph $H_3$ is isomorphic to the triangular prism graph. In Figure 3 we present a drawing for $H_5$ and $H_7$.

**Theorem 3.** Let $q$ be a positive integer with $k = 2q + 1$. Then the $k$-regular graph $H_k$ is not anticonformable.
Proof. Suppose, by contradiction, that $H_k$ is anticonformable. Let $\varphi$ be an anticonformable coloring of $H_k$. Let $C_1, C_2, \ldots, C_{2q+2}$ be the color classes of $\varphi$. From the definition of anticonformable coloring, as $\text{def}(H_k) = 0$ and $|V(H_k)|$ is even, each color class $C_i$ is odd. Hence, every color class is non-empty. Since $U$ and $V$ form a partition into cliques for $V(H_k)$, each color class $C_i$ is singleton. Hence, $|C_1| + |C_2| + \cdots + |C_{2q+2}| = 2q + 2 \neq 2q + 4 = |V(H_k)|$, a contradiction. Therefore, $H_k$ is not anticonformable.

Corollary 1. Let $\lambda$ and $q$ be positive integers with $k = 2q + 1$. If $\lambda$ is odd, then the graph $\lambda K_{2q+2} \cup H_k$ is non-conformable.

Proof. We remark that $\Delta(K_{2q+2}) = \Delta(H_k) = 2q + 1$. Since in any $(2q + 2)$-vertex coloring to $K_{2q+2}$ has each color class singleton, $\lambda K_{2q+2}$ has each color class with size $\lambda$. If $\lambda$ is odd, then $\lambda K_{2q+2}$ is non-conformable. From Theorem 3, $H_k$ has no anticonformable coloring, i.e., any $(2q + 2)$-vertex coloring of $H_k$ has at least one even color class. Hence, any $(2q + 2)$-vertex coloring to $\lambda K_{2q+2} \cup H_k$ has at least one odd color class. Since $|V(\lambda K_{2q+2} \cup H_k)|$ is even and $\text{def}(\lambda K_{2q+2} \cup H_k) = 0$, $\lambda K_{2q+2} \cup H_k$ is non-conformable.

Corollary 2. Let $\lambda$ and $q$ be positive integers with $k = 2q + 1$. If $\lambda$ is odd, then the graph $\lambda K_{2q+2} \cup H_k$ is not Type 1 and non-conformable.

Futher work

In this work, we present the classification of anticonformable $k$-regular graphs with $k$ even and we prove that there is a not anticonformable $k$-regular graph with $k$ odd. Finally, we show an application of the anticonformable definition in order to determine a family of not Type 1 graphs that are non-conformable. As a future work, we aim to characterize, for each integer $k \geq 4$, the family of graphs with maximum degree $\Delta = k$ that are not anticonformable. Finally, we intend to use this result in order to establish the classification of not Type 1 graphs with maximum degree $\Delta = k$ that are non-conformable.
References


