On (acyclic) proper orientations and the cartesian product*

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Abstract. *Given an orientation D of the edges of a simple graph G, the indegree of a vertex $v \in V(G)$, $d^-_D(v)$, is the number of arcs with head in v. Such orientation induces a coloring $\phi(v) = d^-_D(v) + 1$ of G. We say that D is a proper k-orientation if ϕ is a proper $(k+1)$-coloring of G. The proper orientation number of G, denoted by $\overrightarrow{\chi}(G)$, is the least positive integer k such that G admits a proper k-orientation.*
We study a variation of this problem where we consider the orientation D to be acyclic. To the best of our knowledge this is the first article considering this variation. Furthermore, we also study the parameter $\overrightarrow{\chi}$ for graphs obtained by the cartesian product of graphs, introducing the concept of discordant set of proper orientations, that is a set where in different orientations, the same vertex has different indegrees.

1. Introduction

In this work, all graphs are considered to be simple and finite. For basic graph and digraph terminology, see [7, 12]. An *orientation D* of a graph G is an assignment of a direction to each edge of $E(G)$, converting such edge into one of the two possible arcs with the same endvertices. For each $v \in V(G)$, the *indegree* of v, $d^-_D(v)$, is the number of arcs of D with head on v. The orientation D is *proper* if, for every $uv \in E(G)$, $d^-_D(u) \neq d^-_D(v)$. Note that a proper orientation D induces a proper coloring ϕ of G by considering $\phi(u) = d^-_D(u) + 1$. If the maximum indegree of a proper orientation D is at most k, then D is a *proper k-orientation*. The *proper orientation number* of G, $\overrightarrow{\chi}(G)$, is the smallest value k such that G admits a proper k-orientation.

Even though Borowiecki et al. [8] prove that the proper orientation number is well defined in an earlier work, the parameter was defined by Ahadi and Dehghan [1], where the authors note that $\chi(G) - 1 \leq \overrightarrow{\chi}(G) \leq \Delta(G)$. They also prove that it is NP-hard to determine $\overrightarrow{\chi}(G)$, alongside other results. Variations of this parameter have been studied since its introduction. Araujo *et al.* [6] study the parameter $\overrightarrow{\chi}(G)$ for weighted graphs with weight function ω, where the *inweight*, or *weighted indegree*, of a vertex u in a orientation D, $\omega^-(u)$ or $d^-_{(D,\omega)}(u)$, is the sum of the weights of the arcs with head in u. The authors also denote $\overrightarrow{\chi}(G, \omega)$ to be the smallest k such that there exists a proper orientation D of G satisfying $\max_{u \in V(G)} d^-_{(D,\omega)}(u) = k$. In [2], Ahadi *et al.* define the *universal labeling* to be a labeling of the edge set $E(G)$ such that every orientation is proper, when considering the labels as weights and using the same notion of inweight, and they denote by $\overrightarrow{\chi}_u(G)$ the minimum p such that the labels used range from 1 to p. Dehghan and Havet [9] introduce *semi-proper orientations*, where the *semi-proper orientation*

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number of a graph G, $\overrightarrow{\chi_s}(G)$, is the smallest k such that there exists a labeling $\omega : E(G) \to \mathbf{N}$ and proper orientation D of G considering the corresponding inweights satisfying $k = \overrightarrow{\chi}(G, \omega)$. Because it minimum is taken over every possible labeling, then $\overrightarrow{\chi_s}(G) \leq \overrightarrow{\chi}(G)$, as the proper orientation number considers the weights to be 1. It might be counter-intuitive, but using higher labels might decrease the maximum inweight used, while keeping the property of being proper. A curious fact proved by Dehghan and Havet [9] is that there is always an optimal labeling using only labels 1 and 2. This finding is of great interest as it addresses the 1-2-3-conjecture proposed by Karoński *et al.* [11]. Dehghan [10] proposed and studied the *in-out-proper orientation number*, defining the *in-out*-degree of a vertex $v \in V(G)$, by $d^{\pm}(v) = d^{-}(v) - d^{+}(v)$, for an orientation D.

Motivation. Computing $\overrightarrow{\chi}(G)$ is actually an extremely hard problem, as it is NP-hard for sparse graphs like planar bipartite graphs with $\Delta(G) \leq 5$ [4], as well as for chordal graphs [3]. The weighted version is NP-hard even for trees [6]. The proof that $\overrightarrow{\chi}(G) \leq \Delta(G)$ produces an *acyclic* proper $\Delta(G)$-orientation of G. With this information, we address to a constrained version of proper orientations presented in the sequel.

Our contributions. We define the *acyclic proper orientation number* of G, $\overrightarrow{\chi_a}(G)$ as the least k such that there is an acyclic proper orientation D of G satisfying $\max_{v \in V(G)} d^{-}_{D}(v) \leq k$. To the best of our knowledge, this is the first work dealing with this parameter. We prove that this parameter can be arbitrarily far from $\overrightarrow{\chi}(G)$ and also prove upper bounds for it in bipartite graphs and path block graphs in Section 2. In Section 3, we determine $\overrightarrow{\chi}(G \Box H)$ for some graph classes for G and H. For this, we introduce the concept of *discordant set of proper orientations*, that is a set of proper orientations $\{D_1, \ldots, D_q\}$ of G such that $d^{-}_{D_i}(v) \neq d^{-}_{D_j}(v)$, for every $v \in V(G)$ and $i \neq j$.

2. General Bounds on Acyclic Orientations

We know that if T is a tree, any of its orientation is acyclic, hence $\overrightarrow{\chi}(G) = \overrightarrow{\chi_a}(G)$. In [4], the authors prove that $\overrightarrow{\chi}(T) \leq 4$, for any tree T, so it also follows that $\overrightarrow{\chi_a}(T) \leq 4$. A minimization parameter f on a graph class $\mathcal{G}$ is *monotone* if $f(H) \leq f(G)$, for every $H \subseteq G$. In [5], authors prove that the proper orientation number is not a monotone parameter, even for trees. Therefore, $\overrightarrow{\chi_a}$ is also non-monotone.

It is known [7] that every acyclic digraph has a sink. An immediate consequence of that is that there must exist a vertex $v \in V(G)$ such that $d^{-}_{D}(v) = d(v) \geq \delta(G)$, hence $\overrightarrow{\chi_a}(G) \geq \delta(G)$.

2.1. Comparing the proper orientation numbers

In this section we prove that the parameters $\overrightarrow{\chi}(G)$ and $\overrightarrow{\chi_a}(G)$ can be arbitrarily far. We first show that the graph shown in Figure 1 satisfies $\overrightarrow{\chi_a}(G) > \overrightarrow{\chi}(G)$. We then prove that for any positive c there exists a graph G such that $\overrightarrow{\chi_a}(G) \geq \overrightarrow{\chi}(G) + c$.

Proposition 1. *The graph shown in Figure 1 satisfies* $\overrightarrow{\chi_a}(G) > \overrightarrow{\chi}(G)$.

We present the next theorem as a stronger version of Proposition 1. For this theorem, the proof technique involves taking the Cartesian Product of a complete graph with itself.

Proposition 2. *For any given integer c, there exists a graph G such that* $\overrightarrow{\chi_a}(G) \geq \overrightarrow{\chi}(G) + c$.

To prove Proposition 2, we consider $G = K_{2k+1} \Box K_{2k+1}$ and prove that $\overrightarrow{\chi}(G) \leq 3k$ and $\overrightarrow{\chi_a}(G) = 4k$. Considering $n = (2k+1)^2$, this shows us that there exists a graph having n vertices

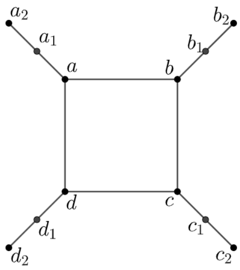

Figure 1. Graph G such that $\overrightarrow{\chi_a}(G) > \overrightarrow{\chi}(G)$

such that $\overrightarrow{\chi_a}(G) \geq \overrightarrow{\chi}(G) + \frac{1}{2}\left(\sqrt{n} - 1\right)$. We investigate if there is a higher function f such that $\overrightarrow{\chi_a}(G) \geq \overrightarrow{\chi}(G) + f(n)$, for some graph G having n vertices.

For the next theorem, we say a *biregular* graph is a bipartite graph for which vertices in the same part have the same degree. For instance, complete bipartite graphs are biregular.

Theorem 3. *Let $G = (X \cup Y, E)$ be a bipartite graph. Denote $k_x = \max\{d_G(x) \mid x \in X\}$ and $k_y = \max\{d_G(y) \mid y \in Y\}$. Then $\overrightarrow{\chi_a}(G) \leq \min\{k_x, k_y\}$. This bound is tight for biregular graphs.*

Since the bound is tight for biregular graphs, notice that for $G = K_{2k+1,2k+1}$, we have that $n = 4k + 2$ and $\overrightarrow{\chi_a}(G) = 2k + 1 = \frac{n}{2}$. Ahadi and Dehghan [1] show that for a $(2k + 1)$-regular bipartite graph G, $\overrightarrow{\chi}(G) = k + 1$. In this instance, it means that $\overrightarrow{\chi}(G) = \frac{n+2}{4}$. Hence, $\overrightarrow{\chi_a}(G) = \overrightarrow{\chi}(G) + \frac{n-2}{4}$, providing us with an even higher function f.

2.2. Block graphs

A *block graph* is a graph where its 2-connected components are cliques, named *blocks*. We say it is uniform (or k-uniform) if every clique has exactly k vertices. A block graph is a *path block graph* if it is connected, each block has at most two cut-vertices and each cut-vertex belongs to exactly two cliques.

In [3], the authors prove that, for $k \geq 3$, a k-uniform block graph G whose blocks have at most two cut-vertices can be properly oriented such that the maximum indegree is at most $k+1$. In their proof, they consider transitive orientations in the cliques, therefore the orientation obtained is acyclic, meaning that $\overrightarrow{\chi_a}(G) \leq k+1$, for k-uniform path block graph G whose blocks have at most two cut-vertices. We now prove a stronger version of this statement, tightening the upper bound and loosening the uniformity condition.

Theorem 4. *Given a path block graph G, let $\omega = \omega(G)$. Then $\overrightarrow{\chi}(G) \leq \overrightarrow{\chi_a}(G) \leq \omega$. These bounds are tight.*

3. Discordant Set of Proper Orientations

Let $G = (V, E)$ be a simple graph. A set of optimal proper orientations $\{D_1, \ldots, D_k\}$ of a graph G is *discordant* if $d^-_{D_i}(v) \neq d^-_{D_j}(v)$, whenever $i \neq j$, for every vertex $v \in V(G)$ and for every $i, j \in \{1, \ldots, k\}$. As an abuse of notation we also say that two distinct orientations D_i and D_j are discordant if $\{D_i, D_j\}$ is discordant. We also define the *proper orientation discordant number* of G, $D_{\overrightarrow{\chi}}(G)$, as the largest integer k such that there exists a discordant set of orientations of G of cardinality k.

As an immediate consequence of this definition, it follows that $D_{\overrightarrow{\chi}}(G) \leq \delta(G) + 1$. Furthermore, the proper orientation discordant number can be directly evaluated for some basic graph classes, such as complete ($D_{\vec{\chi}}(K_n) = n$), path ($D_{\vec{\chi}}(P_n) = 2$) and cycles ($D_{\vec{\chi}}(C_n) = 3$, if 3 divides n and $D_{\vec{\chi}}(C_n) = 2$, otherwise).

We say a digraph $D = (V, A)$ is indegree k-regular if $d^-_D(v) = k$, for every $v \in V$. As an abuse of notation, we that a graph G is indegree k-regular if there exists an orientation D that is indegree k-regular. For instance, it can be proven that K_{2n+1} is n-indegree-regular, for every n.

Theorem 5. *Let G, H be graphs such that $D_{\vec{\chi}}(G) \geq \chi(H)$ and H is indegree k-regular, for some $k \in \mathbb{Z}$. Then $\vec{\chi}(G \square H) \leq \vec{\chi}(G) + k$. Furthermore, $D_{\vec{\chi}}(G \square H) \geq D_{\vec{\chi}}(G)$.*

It follows that if $H = K_{2n+1}$ and $D_{\vec{\chi}}(G) \geq 2n + 1$, then $\vec{\chi}(G \square K_{2n+1}) \leq \vec{\chi}(G) + n$, if $H = C_n$, for $n \geq 4$ even, and $D_{\vec{\chi}}(G) \geq 2$, then $\vec{\chi}(G \square C_n) \leq \vec{\chi}(G) + 1$ and if $H = C_n$, for $n \geq 3$ odd, and $D_{\vec{\chi}}(G) \geq 3$, then $\vec{\chi}(G \square C_n) \leq \vec{\chi}(G) + 1$.

While we stated that all graphs are simple and finite, in the proof presented for Theorem 5 such finiteness is not required. As a conclusion, for an infinite square grid G, it holds that $\vec{\chi}(G) \leq 3$.

Next theorem shows an upper bound for $\vec{\chi}(G \square H)$ where H is a regular bipartite graph.

Theorem 6. *Let G be a graph such that $D_{\vec{\chi}}(G) \geq 2$ and H be a r-regular bipartite graph. for a positive integer r. Then $\vec{\chi}(G \square H) \leq \vec{\chi}(G) + \left\lceil \frac{r}{2} \right\rceil$. Furthermore, $D_{\vec{\chi}}(G \square H) \geq D_{\vec{\chi}}(G)$.*

Next we prove a theorem similar to Theorem 5, but the indegree regularity condition on H is swapped for a similar condition, allowing it to be indegree regular in all but one vertex.

Theorem 7. *Let G, H be graphs and $v \in V(H)$ such that $D_{\vec{\chi}}(G) \geq \chi(H - v)$ and there is an orientation D of H that satisfy (1) $v \in V(H)$ such that $d^-_D(v) = p$; (2) for every vertex $u \in V(H) \setminus \{v\}$, $d^-_D(u) = k \neq p$; and (3) $H - v$ has $d_H(v)$ components. Then $\vec{\chi}(G \square H) \leq \vec{\chi}(G) + \max\{k, p\}$. Furthermore, $D_{\vec{\chi}}(G \square H) \geq D_{\vec{\chi}}(G)$.*

From Theorem 7, it follows that if H is a star graph, then $\vec{\chi}(G \square H) \leq \vec{\chi}(G) + 1$, if $H = P_n$, $n \geq 3$, and $D_{\vec{\chi}}(G) \geq 2$, then $\vec{\chi}(G \square P_n) \leq \vec{\chi}(G) + 1$ and if f H is a tree that is not a star and $D_{\vec{\chi}}(G) \geq 2$, then $\vec{\chi}(G \square H) \leq \vec{\chi}(G) + 1$.

4. Conclusion

In this work, we introduced the concepts of Acyclic Proper Orientations and Discordant Set of Proper Orientations. Regarding the former, we were able to establish a relation between $\vec{\chi}_a$ and $\vec{\chi}$, while providing upper bounds for $\vec{\chi}_a(G)$ for path block graphs and for bipartite graphs. We have shown graphs where $\vec{\chi}_a(G) \geq \vec{\chi}(G) + \frac{n-2}{4}$. We now propose the question of whether the difference $\vec{\chi}_a(G) - \vec{\chi}(G)$ can be improved

Regarding Discordant Set of Proper Orientations, we have shown theorems that allow us to evaluate $\vec{\chi}(G \square H)$, for graphs H satisfying some regularity constraints. Such theorems provided us with bounds for that product when H is a complete having an odd number of vertices, paths, cycles, trees and stars. However, no bound is so far known for when H is a complete graph having an even number of vertices. We leave this as an open problem as well.

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