Bounds on Identifying Codes in the Cartesian Product of a Star and a Path Graph

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Abstract. *In a graph, an identifying code (or ID code, for short) is a dominating set with the property that the closed neighborhood of each vertex in the graph has a distinct intersection with the set. Thus every vertex can be uniquely identified by this intersection. The ID code number of a graph G is the minimum cardinality of an ID code of G and is denoted by $\gamma^{ID}(G)$. We present lower and upper bounds for γ^{ID} in the Cartesian product of star and path graphs.*

1. Introduction

We consider finite, simple, and undirected graphs and use standard notation and terminology. For a graph G, the vertex and edge set are denoted $V(G)$ and $E(G)$, respectively. For a vertex u of G, the *neighborhood*, and the *closed neighborhood* are denoted $N_G(u)$ and $N_G[u]$, respectively. We write $N[v]$ when there is no ambiguity.

Let $C \subseteq V(G)$. We say that C is a *dominating set* if, for each vertex $u \in V(G)$, $C \cap N[u] \neq \emptyset$. A vertex u of C *dominates* a vertex v if either $u = v$, or u is adjacent to v. Two vertices u, v are *separated* by C if $N[u] \cap C \neq N[v] \cap C$; set C is a *separating set* if every pair of distinct vertices of $V(G)$ are separated by C. A subset $C \subseteq V(G)$ is an *identifying code* (ID code, for short) if C is both a dominating set and a separating set of G. The members of C are called *codewords*. Note that a graph has an identifying code if and only if no two vertices have the same closed neighborhood. The ID code number of a graph G is the minimum cardinality of an ID code of G and is denoted by $\gamma^{ID}(G)$.

ID codes were first introduced by [Karpovsky et al. 1998] to model fault-detection problems in multi-processor systems. Since then, ID codes have been studied in many classes of graphs. A detailed list of references on ID codes can be found on Jean's webpage [Jean 2023]. It is algorithmically hard [Auger 2010, Charon et al. 2003] to determine identifying codes of minimum order even for planar graphs of arbitrarily large girth.

For Cartesian products, the problem of finding ID codes has already been studied in a variety of classes, such as paths [Cohen et al. 1999], cliques [Goddard and Wash 2013, Gravier et al. 2008], and the Cartesian product of a path and a clique [Hedetniemi 2016]. Upper and lower bounds for the ID code number of the Cartesian product $G \square K_2$ were given [Rall and Wash 2017]. In the present paper, we further study ID codes focusing on the Cartesian products of star and path graphs, also known by *stacked book* graphs. We present lower and upper bounds for γ^{ID} in these graphs.

2. ID code in $K_{1,n} \square P_m$

We use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n]_0 = \{0\} \cup [n]$. Throughout this section, let $K_{1,n}$ be a star graph of order $n + 1$, $n \geq 2$, and vertex set $V(K_{1,n}) = \{v_0, v_1, \ldots, v_n\}$

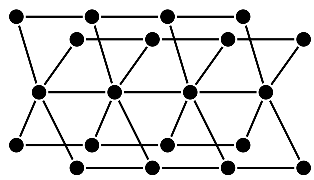


Figure 1. Graph $K_{1,4} \,\square\, P_4$.

such that v_0 is its universal vertex, and let $P_m : u_1u_2 \ldots u_{m-1}u_m$ be a path graph of order m. Since $K_{1,1}$ and $K_{1,2}$ are path graphs, we consider $m \geq 3$.

The Cartesian product of two graphs G and H, denoted by $G \,\square\, H$, is the graph with vertex set $V(G \,\square\, H) = V(G) \times V(H)$ and edge set $E(G \,\square\, H)$ satisfying the following condition: $(u, u')(v, v') \in E(G \,\square\, H)$ if and only if either $u = v$ and $u'v' \in E(H)$ or $u' = v'$ and $uv \in E(G)$. See in Figure 1 a stacked book graph that is the Cartesian product of the star $K_{1,4}$ and a 4-path.

Let $j, k \in [m]$ and $i \in [n]_0$. The $K_{1,n}$-layer (or P_m-layer) is the subgraph of $K_{1,n}\square P_m$ induced by $V(K_{1,n}) \times \{u_j\}$ (or $V(P_m) \times \{v_i\}$) and it is denoted by $K^j_{1,n}$. The two $K_{1,n}$-layers $K^j_{1,n}$ and $K^k_{1,n}$ are adjacent if $u_ju_k \in E(P_m)$, and non-adjacent otherwise. Let $C \subseteq V(K_{1,n}\square P_m)$. We define the function $f_C : [m] \to [n+1]_0$ by $f_C(j) = |C \cap V(K^j_{1,n})|$. For $j \in [m]$, the set $B_j(C)$, a subset of $[n]_0$, is defined as:

$$B_j(C) = \begin{cases} \{i : C \cap \{(v_i, u_1), (v_i, u_2)\} \neq \emptyset\} & \text{if } j = 1, \\ \{i : C \cap \{(v_i, u_{j-1}), (v_i, u_j), (v_i, u_{j+1})\} \neq \emptyset\} & \text{if } 2 \leq j \leq m-1, \\ \{i : C \cap \{(v_i, u_{m-1}), (v_i, u_m)\} \neq \emptyset\} & \text{if } j = m. \end{cases}$$

We give a necessary condition for C to be an ID Code in $K_{1,n}\square P_m$.

Lemma 1 *For $n \geq 2$ and $m \geq 3$, if C is an ID code in $K_{1,n}\square P_m$, then $|B_j(C)| \geq n$, for all $j \in [m]$.*

Proof. Suppose that C is an ID code in $K_{1,n}\square P_m$ and, for a contradiction, $|B_j(C)| < n$, for some $j \in [m]$. So, there exist $i, k \in [n]_0$ with $i \neq k$ such that $\{i, k\} \cap B_j(C) = \emptyset$. Suppose $i = 0$. Thus (v_k, u_j) is not dominated by C. So we may assume that $i \neq 0$. By symmetry, $k \neq 0$. Since C is dominating, $N[(v_i, u_j)] \cap C = N[(v_k, u_j)] \cap C = \{(v_0, u_j)\}$, which contradicts that C is an identifying code. $\square$

Corollary 2 *For $n \geq 2$ and $m \geq 3$, if C is an ID code in $K_{1,n}\square P_m$, then $f_C(1) + f_C(2) \geq n$, $f_C(m-1) + f_C(m) \geq n$ and $f_C(j-1) + f_C(j) + f_C(j+1) \geq n$ for $2 \leq j \leq m-1$.*

The next result is a sufficient condition for a dominating set C to be a separating set in $K_{1,n}\square P_m$.

Proposition 3 *If $C \subseteq V(K_{1,n}\square P_m)$ is a dominating set satisfying the following conditions, then C is an ID code of $K_{1,n}\square P_m$.*

1. *$|B_j(C)| \geq n$, for all $j \in [m]$;*
2. *$\{(v_0, u_1), (v_0, u_m)\} \subseteq C$ and $\{(v_0, u_{j-1}), (v_0, u_{j+1})\} \cap C \neq \emptyset$, for all $j \in \{2, \ldots, m-1\}$;*
3. *If $3 \leq j \leq m-3$ and $(V(K^j_{1,n}) \cup V(K^{j+1}_{1,n})) \cap C = \{(v_0, u_j), (v_0, u_{j+1})\}$, then $\{(v_0, u_{j-1}), (v_0, u_{j+2})\} \cap C \neq \emptyset$;*

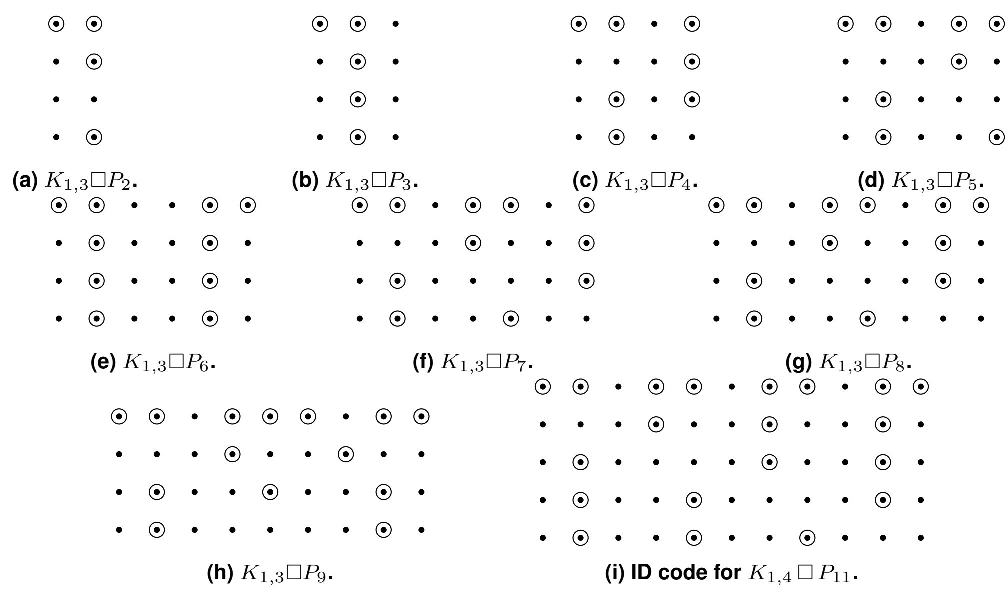
4. *If* $2 \leq j \leq m-2$ *and* $\{(v_0,u_j),(v_0,u_{j+1})\} \cap C = \emptyset$, *then* $f_C(j-1) = f_C(j+2) = n+1$;

Proof. Suppose that C is a dominating set of $V(K_{1,n}\Box P_m)$ as described. Suppose that two vertices (v_x,u_j) and (v_y,u_k) are not separated by C. If $|k-j| \geq 3$, these two vertices have no neighbor in common. So, we assume $j \leq k$ (by symmetry) and $k-j \leq 2$. We consider three cases.

Case 1: $k = j$ (the vertices are in the same $K_{1,n}$-layer). If $x,y \neq 0$, then $N[(v_x,u_j)] \cap C = N[(v_y,u_j)] \cap C = \{(v_0,u_j)\}$. This implies $|B_j(C)| < n$, which is a contradiction. Without loss of generality, suppose $x = 0$. Then, $\{(v_0,u_{j-1}),(v_0,u_{j+1})\} \cap C = \emptyset$, contradicting Condition 2 above.

Case 2: $k = j+1$ (the vertices are in two adjacent $K_{1,n}$-layers). First, consider $x = y = 0$. By Condition 2, $N[(v_x,u_j)] \cap C = N[(v_x,u_{j+1})] \cap C = \{(v_0,u_j),(v_0,u_{j+1})\}$ and then $3 \leq j \leq m-3$, which contradicts Condition 3. If $x \neq 0$ and $x = y$, then one or two of the vertices (v_x,u_j) and (v_x,u_{j+1}) are in C and they do not have another neighbor in C, which implies $\{(v_0,u_j),(v_0,u_{j+1})\} \cap C = \emptyset$. However, this contradicts $N[(v_x,u_j)] \cap C = N[(v_x,u_{j+1})] \cap C$, since by Condition 4 $f_C(j-1) = f_C(j+2) = n+1$. If $x \neq y$ with $x,y \in [n]$, then (v_x,u_j) and (v_x,u_{j+1}) have no common neighbor. Now, without loss of generality, consider $x \in [n]$ and $y = 0$. In this case, $2 \leq j \leq m-2$. The unique neighbor in common with these two vertices is (v_0,u_j). If they are not separated, then $(v_0,u_j) \in C$, $f_C(j+1) = 0$, $(v_0,u_{j+2}) \notin C$ and $x \notin B_j(C)$. However, by Condition 4 $\{(v_0,u_j),(v_0,u_{j+1})\} \cap C = \emptyset$ implies $f_C(j-1) = n+1$, contradicting $x \notin B_j(C)$.

Case 3: $k = j+2$ (the vertices are in non-adjacent $K_{1,n}$-layers). Note that $x = y$, otherwise the vertices have no neighbor in common. Then $N[(v_x,u_j)] \cap C = N[(v_x,u_{j+2})] \cap C = \{(v_x,u_{j+1})\}$, which implies $\{(v_0,u_j),(v_0,u_{j+2})\} \cap C = \emptyset$, contradicting Condition 2 above. □



(a) $K_{1,3}\Box P_2$. **(b)** $K_{1,3}\Box P_3$. **(c)** $K_{1,3}\Box P_4$. **(d)** $K_{1,3}\Box P_5$. **(e)** $K_{1,3}\Box P_6$. **(f)** $K_{1,3}\Box P_7$. **(g)** $K_{1,3}\Box P_8$. **(h)** $K_{1,3}\Box P_9$. **(i) ID code for** $K_{1,4}\Box P_{11}$.

Figure 2. Minimum ID codes for $K_{1,3}\Box P_m$, where $2 \leq m \leq 9$ and for $K_{1,4}\Box P_{11}$.

For ease, we present in Figure 2 the code constructions using a grid in which rows represent the P_m-layers, and the $K_{1,n}$-layers are represented by columns. Vertices in the first row and j-th column are (v_0, u_j), where v_0 is the universal vertex of the star $K_{1,n}$. If vertex (v_i, u_j) is included in the code, then a circle appears in the grid's cell (i, j).

Theorem 4 *For* $m \geq 4$, $\gamma^{ID}(K_{1,3}\Box P_m) \leq \begin{cases} \frac{3m}{2}, & \text{if } m \equiv 0 \pmod 8, \\ \lfloor \frac{3m}{2} \rfloor + 1, & \text{otherwise.} \end{cases}$

Proof. For $2 \leq m \leq 9$, we present the codewords of $K_{1,3}\Box P_m$ in Figures 2(a)-2(h). Let $m = 8q + r$, with $q \geq 1$. We construct the set C as follows. If $r = 0$, we add to C for each block of eight consecutive $K_{1,3}$-layers the codewords presented in Figure 2(g). If $q > 1$ and $r = 1$, we add for each of the first $q - 1$ blocks of eight consecutive $K_{1,3}$-layers the code presented in Figure 2(g) and for the last block of nine consecutive $K_{1,3}$-layers the code presented in Figure 2(h). If $q > 1$ and $r \in \{2, \ldots, 7\}$, we add for each of the first q blocks of eight consecutive $K_{1,3}$-layers the code presented in Figure 2(g) and for the last block of r consecutive $K_{1,3}$-layers the code presented in the respective Figure 2(a)-2(f). By Proposition 3, C is an ID code. By inspection, it is possible to verify that C has the described cardinality. $\Box$

The next result improves the bound presented in Corollary 2.

Proposition 5 *For* $m \geq 6$, $\gamma^{ID}(K_{1,3}\Box P_m) \geq 4\lfloor \frac{m}{3} \rfloor - 1$.

Proof. (Idea) Let $j \in \{2, \ldots, m-1\}$. By Corollary 2, $f_C(j-1) + f_C(j) + f_C(j+1) \geq 3$. Suppose that C is a minimum identifying code of $K_{1,3}\Box P_m$ and for some $j \in \{2, 3, \ldots, m-4\}$, $f_C(j-1) + f_C(j) + f_C(j+1) = 3$. We show that this implies $f_C(j+2) + f_C(j+3) + f_C(j+4) \geq 5$. Note that if $f_C(j) = 0$, since C is dominating, $f_C(j-1) + f_C(j+1) \geq 4$ and we are done. So, we assume $f_C(j) \neq 0$. By symmetry, we consider $j \leq m - 4$. We have three cases: Case 1: $f_C(j) = 1$, Case 2: $f_C(j) = 2$, and Case 3: $f_C(j) = 3$. In every case, we show that either C is not a separating set or $f_C(j+2) + f_C(j+3) + f_C(j+4) \geq 5$, which, on average, gives the result we stated. $\Box$

Finally, we present ID codes for $K_{1,n} \Box P_m$ for an arbitrary n and specific m, which give us an upper bound for γ^{ID} in these cases.

Theorem 6 *If* $n \geq 2$ *and* $m \geq 5$ *is a multiple of* $3n - 1$, $\gamma^{ID}(K_{1,n} \Box P_m) \leq \frac{m(n^2+n)}{3n-1}$.

Proof. First, we show how to recursively construct a set $C_{n,m}$ in $K_{1,n}\Box P_m$, with $n \geq 2$ and $m = 3n-1$. Let the set $C_{2,5} = \{(v_0, u_1), (v_0, u_2), (v_0, u_4), (v_0, u_5), (v_1, u_4), (v_2, u_2)\}$. For $n \geq 3$, $C_{n,m} = C_{n-1,m-3} \cup \{(v_i, u_{m-1}) \mid 0 \leq i \leq n-1\} \cup \{(v_0, u_m)\} \cup \{(v_n, u_j) \mid j \equiv 2 \pmod 3$ and $j \leq m - 3\}$. See in Figure 2(i) a construction for $K_{1,4} \Box P_{11}$. Note that $|C_{2,5}| = 6$ and for $n \geq 3$ and $m \geq 8$, $|C_{n,m}| = |C_{n-1,m-3}| + n + \lceil \frac{m}{3} \rceil = n^2 + n$. Let $q = \frac{m}{3n-1}$. When $q = 1$, we do $C = C_{n,m}$ and when $q > 1$, we add in C for the next $q - 1$ blocks of adjacent $K_{1,n}$-layers the code for $C_{n,3n-1}$. It follows that $|C| = \frac{m(n^2+n)}{3n-1}$ and, by Proposition 3, C is an ID code in $K_{1,n} \Box P_m$. $\Box$

3. Concluding remarks

We present lower and upper bounds for γ^{ID} in the Cartesian product of a star and a path graph. The next steps are to generalize the presented results for $K_{1,3} \Box P_m$ and to determine closed formulas for the general case.

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