An Introduction to the Complexity Class of Pure Nash Equilibrium

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Abstract. *Taxonomy of problems in Computer Science has been typically done with formulations as decision problems. This approach is inadequate for many search problems of interest when the structure of the instance in itself guarantees the existence of a positive certificate. In this paper, we provide an introduction to the class of problems* PLS *and its connections with finding Pure Nash Equilibrium in Congestion Games.*

1. Introduction

Definition. A *search problem* L is defined by a relation $R_L \subseteq \{0,1\}^* \times \{0,1\}^*$ such that $(x, y) \in R_L$ if and only if y is a solution to x. A search problem L is called *total* if and only if for all x there exists y such that $(x, y) \in R_L$.

Definition. A problem L is said to be in FNP (*Function Nondeterministic Polynomial*) if and only if there exists a polynomial-time algorithm $A_L(\cdot,\cdot)$ and polynomial function $p_L(\cdot)$ such that

i. $\forall x, y \; A(x, y) = 1 \iff (x, y) \in R_L$
ii. $\forall x : \exists y \; : (x, y) \in R_L \implies \exists z : |z| \leq p_L(|x|) \wedge (x, z) \in R_L$.

Definition. A problem L belongs to FP (*Function Polynomial*) if there is an algorithm A that given any instance $x \in L$, A finds y such that $(x, y) \in R_L$ or reports that none exists with a number of steps of computation bounded by a polynomial in the length of x.

Definition. (*Total* FNP) TFNP $= \{L \in \text{FNP} \mid L \text{ is total}\}$.

TFNP is a difficult class to study as a whole, mainly because the argument behind the existence of a solution for different problems might differ. Different arguments providing the existence of solutions call for an analog of NP-completeness for other domains of problems. Next, we discuss the class that relates to the concept of pure Nash equilibrium in Game Theory.

2. Polynomial Local Search

In many optimization problems finding a global optimum can be difficult enough to the point that an exhaustive search is the only known way to find the desired certificate. For these problems, we lower the expectations and, instead of striving for the global optimum, we content with a local one, according to a neighborhood criterion. An example is the Maximal Cut, shortened to MAX-CUT.

Definition. Let $\mathcal{G} = (V, E)$ be a graph, a cut is a partition of V into subsets A and A^c. We call the sum of capacities of edges connecting vertices from A to A^c the *cut capacity*.

> **Maximal Cut Problem.** Given a graph $\mathcal{G} = (V, E)$, find a cut such that its capacity is at least the size of any other, i.e., a maximal cut.

The decision version of MAX-CUT is NP-*complete* [Garey and Johnson 1990], therefore a polynomial-time algorithm in the size of $\mathcal{G}$ that finds an optimal solution inexists unless P = NP. *Local search* is an heuristic utilized to provide feasible solutions for many NP-hard problems [Papadimitriou and Steiglitz 1998], including the MAX-CUT. It starts with an arbitrary feasible solution and increments the objective function according to the available neighborhood[1][2].

For MAX-CUT, we consider the *neighborhood* concept in that two solutions differ by only a single vertex placement, so moving to A^c a vertex v currently in A and vice versa is a *local movement*. The cost of the solution (when moving a vertex from A to A^c) increments by an additive factor

$$\sum_{u \in A : (u,v) \in E} c_{u,v} - \sum_{u \in A^c : (u,v) \in E} c_{u,v}. \tag{2.1}$$

If this difference is positive, then the local movement is *valid*. For the general case with nonnegative edge costs, the algorithm may take exponential time in the number of vertices. Figure 1 illustrates a local maximum for a MAX-CUT instance with parts being denoted by vertices' colors.

For local search, if we interpret a given feasible solution as a vertex on a graph and each neighboring feasible solution as vertices connected to it, where the direction of the edges is given according to valid local movements, we then have a directed acyclic graph (*dag*). The vertices representing local optima are *sinks* – vertices with no outgoing edge – and the initial feasible solution is a *source* – vertex with no incoming edge. Such a graph is sometimes called *transition graph* in the literature, and for an instance of MAX-CUT, each vertex in its transition graph is a possible cut.

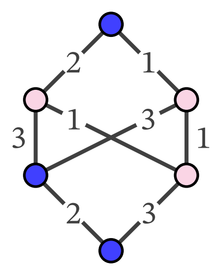

Figure 1. Local optimum of a maximal cut instance: v in A are colored in dark blue, v in A^c colored in pink.

> **Sink of dag.** Given a source vertex in a directed acyclic graph (*dag*) $\mathcal{G}$, find a vertex with no outgoing edges.

[1] We call *neighborhood* the set of solutions that differ from a given solution by a minimal possible extent.

[2] The general procedure for a local search is composed of three implicit subroutines: verification of feasibility of a solution; inspection of neighboring solutions; and computation of a solution's cost. All assumed to run in polynomial time in the length of the input.

For each vertex in $\mathcal{G}$ there are two possible states when it comes to making a cut: it either belongs to the part A or not, making a total of $2^{|V|}$ possible cuts. Thus we say the graph defined for SINK-OF-DAG is *exponential in the description of the problem* MAX-CUT and local search may take exponential time in the size of $\mathcal{G}$.

A problem L is said to be in PLS (*Polynomial Local Search*) if there is a polynomial-time algorithm that reduces L to a SINK-OF-DAG instance. Since, by definition, a connected dag must have at least one source and at least one sink, a solution is guaranteed to exist. Thus, PLS $\subset$ TFNP. It turns out that SINK-OF-DAG is polynomial-time reducible to MAX-CUT [Schäffer and Yannakakis 1991], making it PLS-complete.

2.1. Congestion Games

Congestion games are a special type of game where each player has her/his strategy attached to a finite set E of resources. Each resource has an associated cost, which takes the number of players choosing it as a parameter. The payoff of a player is a function over the subset of resources she/he has taken as strategy. Whenever speaking about games, the assumption of rationality is taken, that is: each player aims to optimize her/his utility function.

We denote the strategy player i picks by $s_i \in S_i$, where S_i is the set of all available strategies for i, and collectively refer to strategies chosen by other players as $\mathbf{s}_{-i}$, so that the strategy profile is $\mathbf{s} = (s_i, \mathbf{s}_{-i})$.

Definition. (Pure Nash Equilibrium, [Turocy and von Stengel 2003]) A strategy profile $\mathbf{s} = (s_i, \mathbf{s}_{-i})$ of a cost-minimization game with cost function C_i for each player $i \in N$ is a *pure Nash equilibrium (PNE)* if $C_i(\mathbf{s}) \leqslant C_i(s_i', \mathbf{s}_{-i})$, for all i and every unilateral deviation $s_i' \in S_i$. In this case, we say that s_i is a *best response* to $\mathbf{s}_{-i}$.

Theorem 2.1. ([Rosenthal 1973]) Every congestion game has at least one Pure Nash Equilibrium.

Proof. Let $\mathbf{s}$ be an assignment of strategies for each player, c_e be the cost function for resource e and f_e be the number of players picking resource e.

We define the *potential function* $\phi(\mathbf{s}) = \sum_{e\in E} \sum_{i=1}^{f_e} c_e(i)$. Moreover, under unilateral deviation, we have that

$$\begin{aligned}
\phi(s_i', \mathbf{s}_{-i}) - \phi(\mathbf{s}) &= \sum_{e\in s_i'\setminus s_i} c_e(f_e+1) - \sum_{e\in s_i\setminus s_i'} c_e(f_e) \\
&= \sum_{e\in s_i'} c_e(f_e') - \sum_{e\in s_i} c_e(f_e) \\
&= C_i(s_i') - C_i(s_i). \qquad (2.2)
\end{aligned}$$

Here f_e' is the number of players picking resource e for strategy profile $(s_i', \mathbf{s}_{-i})$. The change in ϕ is the change in player i's cost when deviating. Since there is a finite number of assignments of congestible resources, there is a minimum for ϕ. ■

Congestion games can thus be shown to belong to PLS by computing a local minimum through means of *best-response dynamics*, the analogous for local search in

Game Theory, where the neighborhood is defined by the unilateral deviations that benefit the deviator's payoff. [Fabrikant et al. 2004] first sketched a proof that congestion games are PLS-complete through a reduction from MAX-CUT. We present the proof in the following.

Theorem 2.2. All problems in PLS are polynomial-time reducible to the problem of finding a pure Nash equilibrium in congestion games.

Proof. Let $\mathcal{G} = (V, E)$ be a MAX-CUT instance with each edge $e \in E$ having capacity w_e. We describe an instance of a cost-minimization congestion game for which finding pure Nash equilibria is equivalent to finding a local maximal cut for $\mathcal{G}$. The corresponding instance of the congestion game has a player for each vertex of $\mathcal{G}$ and two resources r_e and $\bar{r}_e$ for each edge e of G. Each player v has two strategies of resources: $s_v = \{r_e\}$ for all r_e such that $v \in e$ and $\bar{s}_v = \{\bar{r}_e\}$ for all $\bar{r}_e$ such that $v \in e$. Thus, each resource associated with an edge e is available only for the two players corresponding to its endpoints. Moreover, let $n = |V|$, since each player has two strategies, there is a total of 2^n strategy profiles, in perfect correspondence with the $2^{|V|}$ possible cuts for $\mathcal{G}$.

Let k be either $\bar{r}_e$ or r_e and f_k be the number of players whose strategy includes k, the cost functions are defined as

$$c_k(f_k) = \begin{cases} 0 \text{ if } f_k = 1, & \text{else } w_e \end{cases}.$$

Fix a cut (A, A^c). We establish the following bijection: a player v who chooses its strategy s_v corresponds to a vertex v in part A, and v who chooses strategy $\bar{s}_v$ corresponds then to a vertex v in part A^c. Let $C(A, A^c)$ be the capacity of the cut. The game has potential function $\phi(\mathbf{s}) = \sum_{e \in E} w_e - C(A, A^c)$, so maximizing the cut capacity is equivalent to minimizing this potential function. Besides, one can check that under unilateral deviation, the change in ϕ is equal to (2.1), appropriately corresponding pure Nash equilibria – local minima in the potential function – to local maxima in MAX-CUT. ■

In Figure 1, vertices colored in dark blue correspond to players choosing their respective strategy sets s_v, and vertices colored in pink correspond to players choosing their respective strategy sets $\bar{s}_v$. An edge e between vertices of the side of the cut contributes with w_e to each of its corresponding players' payoff. Edges connecting vertices from different parts contribute with zero to their costs. When a player deviates to a beneficial strategy, her/his previous payoff contributes with the first summation on (2.1), whilst the new payoff contributes with the second one. Since we assume the deviation to be beneficial, the first summation is greater than the second one, characterizing a valid local movement.

3. Conclusion

Whether or not FP is a proper subset of PLS is still an open-problem in Computer Science. Even if a proof could show that PLS = FP, a local search heuristic can potentially run in exponential time, so an efficient algorithm to find a PNE is yet to be seen, if one such algorithm exists. For sources in Algorithmic Game Theory, we recommend [Nisan et al. 2007], in particular Chapter 18, which is the most relevant this work.

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