Minimum Diameter Trees Subject to a Budget Constraint

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Abstract. *Formulations are proposed for problems that ask for minimum diameter trees of an undirected graph, subject to an upper bound on the sum of their edge costs. The problems, namely the Budget Restricted Minimum Diameter Spanning Tree, the Budget Restricted Minimum Diameter Steiner Tree, and the Budget Restricted Minimum Diameter Terminal Steiner Tree, represent challenging extensions for three intensively investigated NP-hard problems. Furthermore practical applications for these extensions naturally carry over from those for the problems they originate from. Three distinct types of formulations are proposed here for each of the previously indicated problem extensions. Computational results for these formulations will be presented separately.*

1. Introduction

Let $G = (V, E)$ be a simple undirected connected graph defined by a set of vertices V and a set of edges E. A *path* is a finite sequence of the edges of E connecting a sequence of distinct vertices of V. The *length* of a path corresponds to the number of edges it contains, and a path between $s \in V$ and $t \in V \setminus s$, i.e., a s-t-path of G, has the shortest possible length if there exists no other s-t-path with fewer edges. Finally, the *diameter* of G is the length of the longest shortest path connecting two distinct vertices of G.

Assuming that costs are associated with the edges of $G = (V, E)$ and a number $B \in \mathbb{N}$ is given, the Budget Restricted Minimum Diameter Spanning Tree Problem (BSpT) asks for a minimum diameter spanning tree $T = (V_T, E_T)$ with a sum of edge costs not exceeding B. BSpT was first proposed by [Plesnik 1981], who also proved that its decision version is NP-complete. Apparently, no mathematical formulation or exact solution algorithm for BSpT currently exists and one will be proposed here.

The Steiner Tree Problem in Graphs (STPG) (see [Maculan 1987, Goemans and Myung 1993]), an intensively investigated NP-Hard problem, is a natural candidate for inclusion into our budget-diameter framework. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, STPG asks for a tree $T = (V_T, E_T)$ that spans all vertices of S, i.e., a *Steiner tree* of G, with a sum of edge costs as small as possible. Vertices in S are called *mandatory* or *terminals* while those in $V \setminus S \neq \emptyset$ are the *optional* ones. Moreover, vertices in $V_T \cap (V \setminus S)$ are called *Steiner vertices*. STPG applications are covered in detail in the literature (see in particular [Ivanov et al. 2003, Cheng et al. 2004, Robins and Zelikovsky 2008, Smith and Gross 1982, Mondaini 2008, Smith 1998]).

When the goal is simply to obtain a Steiner tree with minimum diameter, the problem is called the Minimum Diameter Steiner Tree Problem (DSTPG). This restricted form of STPG is thoroughly invested [Marathe et al. 1998]. In particular, its NP-hardness and inherent difficulty of approximation within a logarithmic factor. [Ding and Qiu 2014], in turn, investigate some particular extension of DSTPG. Finally, an upper bound on the sum of the edges costs of a STPG tree, one obtains the Budget Restricted Minimum Diameter Steiner Tree Problem (BStT), which is proposed and investigated here.

Moving one step further, we also investigate a problem that originates from a restricted version of STPG. Specifically, one where all leaves of T, or its vertices with degree 1, must belong to S, in what ammounts to a *Full Steiner Tree* (FST) [Lu et al. 2003]. Finding an FST with the minimum sum of edge costs is an NP-Hard problem known as the Terminal Steiner Tree Problem (TSTP) [Lin and Xue 2002]. Conversely, when FSTs must have the smallest feasible diameter possible, it becomes the Minimum Diameter Terminal Steiner Tree Problem (DTSTP). Extensive research by [Ding and Qiu 2014] has founded on DTSTP. By imposing a budget constraint on DTSTP, we introduce the Budget Restricted Minimum Diameter Terminal Steiner Tree Problem (BTStT), which is also explored in this study. The BStT and BTStT problems are new additions to the academic literature. Both BStT and BTStT, as novel problems which are conjectured to be NP-Hard, given the complexity of their budget free counterparts.

2. Mathematical Formulation

We suggest three distinct types of formulations for BSpT, BStT, and BTStT each. However, due to space limitations, we will present here just one of them.

The formulations are based on two key properties for trees: the *Central Vertex Property* (CVP) and the *Central Edge Property* (CEP), as described in [Handler 1973]. Starting with an undirected graph $G = (V, E)$, as previously defined, we associate with it an artificial vertex r and a directed graph $D = (V', A)$. In doing so, $V' = V \cup \{r\}$ and $A = \{(i, j), (j, i) : e = \{i, j\} \in E\} \cup \{(r, i) : i \in V\}$, are the resulting vertex and edge lots. Costs for the arcs of D are defined as follows: $c_{ij} = c_{ji} = c_e$, where $e = \{i, j\} \in E$, and $c_{ri} = 0$, for all $i \in V$.

In the context of $D = (V', A)$, our objective is to discover a minimum diameter arborescence $T = (V_T, A_T)$ rooted at r. This arborescence directly corresponds to the minimum diameter tree sought for $G = (V, E)$. To account for diameter parity, we use a binary variable z, where $z = 1$ implies odd diameters and $z = 0$ implies otherwise. Moreover, we introduce a non-negative variable $s \in \mathbb{R}$ to represent the actual diameter of T. Specifically, s represents the length of the longest path in T between the artificial vertex r and a leaf of T.

An important parameter in our formulations is an upper bound L on the length of the paths originating from r. Such a bound might be obtained through properly defined heuristics or by simply setting $L = \left\lceil \frac{|V|}{2} \right\rceil$. Our formulations ensure that the length of any path in T does not exceed L, whether it originates from an unidentified central vertex in V or from the endpoints of an unidentified central edge in E. As we shall see, the central vertex option will apply to trees with even diameters while the central edge one will apply to those with odd diameters.

Objective functions are defined as $2(s - 1) + z$ and correspond to the sum of the lengths of the two longest paths of T, originating from either the central vertex or the extremities of the central edge, whatever applies. As we shall see, by minimizing $2(s - 1) + z$ one will naturally obtain a desired minimum diameter tree for G.

Our formulations simultaneously assign variables to both the edges of $G = (V, E)$ and the arcs of $D = (V', A)$. Additionally, these formulations combine the properties of CVP and CEP specifically adapted to the directed graph $D = (V', A)$. Finally, the optimal r-rooted arborescences they return will necessarily have either one or two arcs pointing outwards of r. One arc when their corresponding optimal trees of G have odd diameter and two arcs when their diameter is even.

The variables $\{x_{ij} \in 0, 1 : (i, j) \in A\}$ denote the arcs of the resulting r-rooted arborescence, with $x_{ij} = 1$ if arc $(i, j) \in A$ is included, and $x_{ij} = 0$ otherwise. Variables $\{w_e \in 0, 1 : e \in E\}$ identify the central edge in trees with odd diameter, with $w_e = 1$ if $e \in E$ is central, and $w_e = 0$ otherwise. If $e = \{i, j\} \in E$ is central, $x_{ri} = x_{rj} = 1$ must hold. The auxiliary variable $z \in \{0, 1\}$ indicates the diameter parity, with $z = 1$ for odd and $z = 0$ for even.

Bearing in mind the previous variables and explanations, a combined formulation using CVP and CEP is given by a polyhedral region $\mathcal{R}c$ defined as follows:

$$\sum_{j \in V} x_{rj} = z + 1 \tag{1}$$

$$\sum_{e \in E} w_e = z \tag{2}$$

$$w_e \leq x_{ri}, \qquad e = \{i, j\} \in E, i < j \tag{3}$$

$$w_e \leq x_{rj}, \qquad e = \{i, j\} \in E, i < j \tag{4}$$

$$x_{ij} \in \{0, 1\} \qquad \forall (i, j) \in A \tag{5}$$

$$w_e \in \{0, 1\}, \qquad \forall e = \{i, j\} \in E \tag{6}$$

$$z \in \{0, 1\}. \tag{7}$$

When $z = 0$, constraint (1) ensures that there can only be a single arc emanating from r, characterizing the even case. Conversely, when $z = 1$, exactly two arcs must point away from r, one towards each end of the edge selected in (2), representing the odd case. Constraints (3) and (4) then stipulate that, given a chosen edge $e = \{i, j\} \in E$, the arcs (r, i) and (r, j) must necessarily be included in the solution. Furthermore, domains of the variables denoting arc inclusion (5), central edge identification (6), and diameter parity (7) are affirmed.

In order to obtain a formulation for BSpT, we closely follow a formulation proposed in [Dos Santos et al. 2004] for the Diameter Constrained Minimum Spanning Tree (DCMSTP). Such a formulation imposes solution connectivity through a lifting of the classical Miller-Tucker and Zemlin (MTZ) inequalities [Miller et al. 1960] (see [Desrochers and Laporte 1991, Dos Santos et al. 2004] for the lifting we implemented), described here through variables $u_i \in \mathbb{R}_+, i \in V'$. Accordingly, a formulation for BSpT is thus given by

$$
\begin{array}{llll}
\text{Min} & 2(s-1)+z & & (8)\\
\text{s.a} & \sum_{(i,j)\in A} x_{ij} = 1 & j \in V & (9)\\
& u_i - u_j + (L+1)x_{ij} + (L-1)x_{ji} \leq L & (i,j) \in A & (10)\\
& s \geq u_i & i \in V & (11)\\
& (\mathbf{x}, \mathbf{w}, z) \in \mathcal{R}_c & & (12)\\
& \sum_{(i,j)\in A} c_{ij}x_{ij} + \sum_{e\in E} c_e w_e \leq B & & (13)\\
& 0 \leq u_i \leq L+1 & \forall i \in V' & (14)\\
& s \geq 0. & & (15)
\end{array}
$$

Constraints (9) mandate the inclusion of all vertices from D in the tree. Constraints (10) and (14) limit the path length from the root to any vertex in V to a maximum of $L+1$ edges. These constraints, known as MTZ, effectively prevent the formation of cycles by disallowing the path from looping back onto itself. Equations (11) establish a minimal bound for s, corresponding to the path lengths from r to each vertex $i \in V$, which is minimized in our objective function. The $\mathcal{R}_c$ property is embedded via (12). Constraint (13) ensures the total tree cost does not surpass B.

We will now elucidate the adaptation of our BSpT formulation to BStT. Initially, we adjust constraints (9) to apply solely to vertices in S_T. This ensures that a feasible BStT solution requires inclusion of only the mandatory vertices in D, not all of them.

Concluding our formulation of BStT, we enforce connectivity between root vertex r and the mandatory vertices of D with the well known cutset-type inequalities [Dantzig et al. 1954]. In order to do so, denote by $[V' \backslash C, C]$ a cutset where C is a suitable set obtained through minimum cut or maximum flow methods such that $r \in V' \backslash C$ and $C \cap S_T \neq \emptyset$. Our cutset inequalities are then described as

$$
\sum_{i\in V'\backslash C, j\in C} x_{ij} \geq 1 \qquad r \in V'\backslash C, C \cap S_T \neq \emptyset. \tag{16}
$$

Finally, in order to formulate BTStT, a very straightforward restriction must be imposed on our formulation to BStT. Namely, one that enforces all mandatory vertices, i.e., those in D, to define leaves of the feasible trees.

3. Conclusions

The formulation of BSpT, BStT, and BTStT hinges on accurately representing the diameter parity of feasible trees, a task that is not straightforward. We have addressed this challenge indirectly by proposing a polyhedral region and integrating it into existing DCMSTP formulations. In this context, we have elaborated on formulations predicated on lifted MTZ inequalities. Two additional frameworks, drawing upon the formulations of [Gouveia and Magnanti 2003] and [Gouveia et al. 2011], will be expounded upon in the presentation.

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