Infinite families of Kochol superpositions of Goldberg with Blowup snarks are Type 1*

Giovanna A.B. Penao[1], **Miguel A. D. R. Palma**[1], **Simone Dantas**[1], **Diana Sasaki**[2]

[1]IME, Universidade Federal Fluminense, Brazil.

[2]IME, University of the State of Rio de Janeiro, Brazil.

arelis_giovanna@id.uff.br, miguedrp@gmail.com

sdantas@id.uff.br, diana.sasaki@ime.uerj.br

Abstract. *A q-total coloring of a graph G is an assignment of q colors to the vertices and edges of G, so that adjacent or incident elements have different colors. The Total Coloring Conjecture (TCC) asserts that an optimum total coloring of G has at least $\Delta+1$ and at most $\Delta+2$ colors. We determine that all members of new infinite families of snarks obtained by the Kochol superposition of Goldberg with Blowup snarks are Type 1.*

Resumo. *Uma q-coloração total de G é uma atribuição de q cores aos vértices e arestas de um grafo G, de forma que elementos adjacentes ou incidentes tenham cores diferentes. A Conjectura da Coloração Total (TCC) afirma que uma coloração total ótima de G é pelo menos $\Delta + 1$ e no máximo $\Delta + 2$ cores. Determinamos que todos os membros de novas famílias infinitas de snarks obtidos pela superposição Kochol de snarks de Goldberg com snarks Blowup são Tipo 1.*

1. Introduction

The study of *snarks*, a specific class of cubic graphs, originated from the Four Color Conjecture. These cyclically 4-edge-connected cubic graphs pose a challenge in edge coloring area, as they cannot be colored with merely three colors, ensuring that no two incident edges share the same color. The Petersen graph stands as the earliest and smallest example of a snark. Isaacs [7] expanded the snark family by introducing additional examples such as the Flower and Loupekine snarks, along with the dot product operation, leading to the creation of an infinite number of snarks.

This paper focuses on finite, undirected, and simple graphs represented as $G = (V, E)$, where V denotes the set of vertices and E the set of edges. The *maximum degree* of G is denoted as $\Delta(G)$ or simply Δ. The concept of a q-total coloring for G involves assigning q colors to its vertices and edges in a manner that ensures adjacent or incident elements have distinct colors. The *total chromatic number* of G, denoted as $\chi''(G)$ or simply χ'', represents the minimum value of q required for a q-total coloring of G. An

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equitable q-total coloring is a q-total coloring where the color class cardinalities differ by at most one.

The *Total Coloring Conjecture* (TCC) states that the total chromatic number of a graph G ranges from $\Delta + 1$ to $\Delta + 2$ (Behzad [1], Vizing [13]). This conjecture led to the characterization of graphs into *Type 1* and *Type 2*, depending on whether χ'' equals $\Delta+1$ or $\Delta+2$. Despite verification for specific graph families, the TCC remains an open problem for various graph classes but, particularly, for cubic graphs Rosenfeld [11] and Vijayaditya [12] independently verified that $4 \leq \chi''(G) \leq 5$.

The *girth* of G, which is the length of the shortest cycle contained in G, is a crucial parameter. In 2003, Cavicchioli et al. [4] demonstrated, with computer assistance, that snarks with girth at least 5 and fewer than 30 vertices are Type 1. They posed the problem of finding the smallest Type 2 snark with girth at least 5. This question was addressed in 2011 when Campos, Dantas, and Mello [3] provided an equitable total coloring for certain snarks with girth 5.

In 1996, the concept of the *superposition* construction introduced by Kochol [8] yielded infinite snark families with large girth. Subsequently, in 2016, Hägglund [6] defined two additional infinite snark families - *Blowup* and *SemiBlowup* snarks. In 2022, Palma et al. [10] demonstrated that SemiBlowup snarks have equitable total colorings with 4 colors, contributing to the exploration of Type 1 graphs with girth at least 5. Later, Palma et al. [9] also determined that the Kochol superposition of Goldberg with Semi-blowup snarks is Type 1.

In this work, we contribute to the study of superpositions with Hägglund snarks presenting Type 1 new infinite families of snarks obtained by the Kochol superposition of Goldberg with t-Blowup snarks. Moreover, these results contribute to a question posed by Brinkmann et al. [2] by presenting negative evidence about the existence of Type 2 cubic graphs with girth at least 5.

2. Kochol superposition of Goldberg with Blowup snarks

Goldberg snarks were introduced by Goldberg [5] in 1981 whose construction is obtained recursively from linking basic blocks. In Figures 1(a) and 1(b), we depict the first Goldberg snark G_3 and the *link semi-graph* $\mathcal{L}_i$. The second member of this family, Goldberg snark G_5 (see Figure 1(c)), is obtained by deleting the edges from G_3 depicted as dashed lines in Figure 1(a), adding corresponding semiedges, and making their corresponding junction with the semiedges of the link semi-graph $\mathcal{L}_5$. All subsequent members in this family, denoted as G_i with odd $i \geq 7$, are similarly constructed.

Campos, Dantas and Mello [3] showed that Goldberg snarks admit 4-total colorings using the colorings of the Goldberg snark G_3 and of the link graph $\mathcal{L}_i$, odd $i \geq 5$.

Theorem 1 ([3]). *Each Goldberg snark G_i, with odd $i \geq 3$, is Type* 1. □

The Blowup family of snarks was introduced in 2016 by Hägglund [6]. Let S_p' be the semi-graph depicted in Figure 2(a). The *t-Blowup* is a snark constructed by connecting $t \geq 5$ copies of semi-graph S_p', with $1 \leq p \leq t$ (as a cycle).

Palma et al. [10] presented equitable 4-total colorings for these graphs proving that these graphs are Type 1.

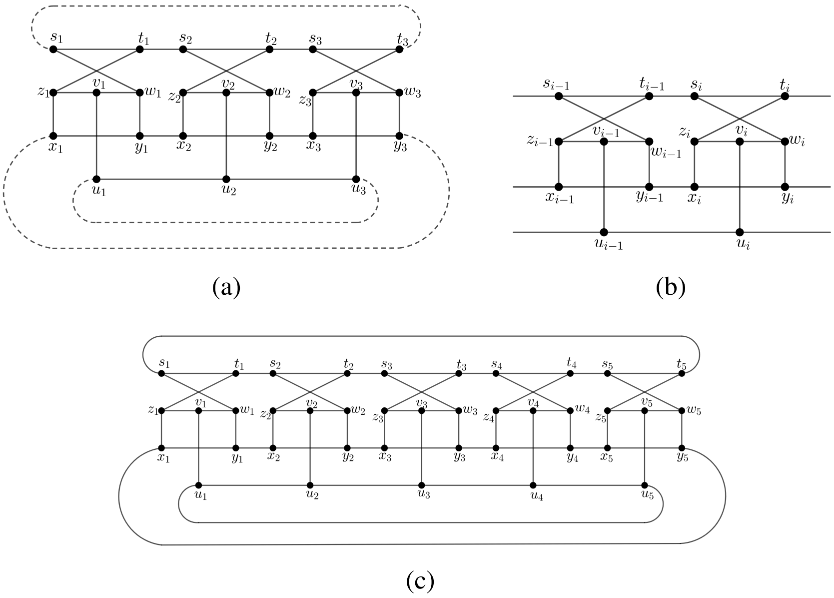

Figure 1. (a) Goldberg snark G_3; (b) Link semi-graph $\mathcal{L}_i$, with odd $i \geq 5$; and (c) Goldberg snark G_5.

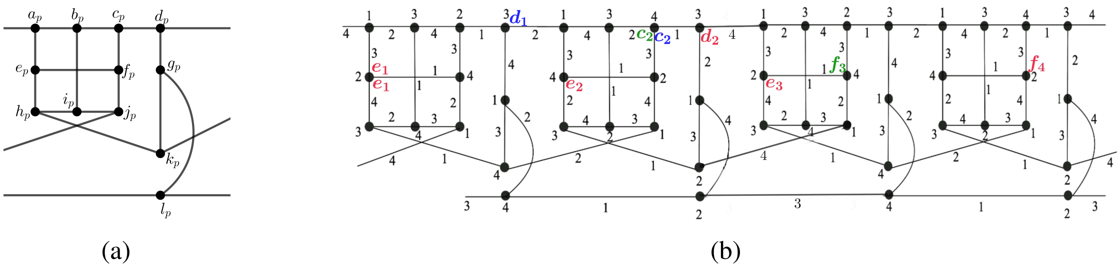

Figure 2. (a) Semi-graph S_p' of t-Blowup snarks; (b) 4-total coloring of semi-graph B_4' that appears in every t-Blowup, with $t \geq 6$. Each pair of vertex labels highlighted with the same color represents the pair that is removed to construct the superedge ξ'.

Theorem 2 ([10])**.** *Each t-Blowup snarks, with $t \geq 5$, are Type 1.*

Next, we define the *Kochol superposition* construction as presented in [8]. Given a cubic *semi-graph* $M(V, E, S_E)$, with $S_E \neq \emptyset$, the set S_E of semiedges is partitioned into q pairwise disjoint nonempty sets $Q_1, Q_2, \ldots, Q_q$ such that $|Q_i| = k_i$ with $i = 1, 2, \ldots, q$ and $\sum_{i=1}^{q} k_i = |S_E|$. Following the notation of [8], we call the sets Q_i *connectors*, and denote the semi-graph M by (k_1, k_2, ..., k_q)-semi-graph M.

A *superedge* ξ is a semi-graph with two connectors, and a *supervertex* ϑ is a semi-graph with three connectors. We consider the following semi-graphs depicted in Figure 3:

i. $(3, 3)$-semi-graph M' (superedge) is obtained by removing two nonadjacent vertices v_1 and v_2 from a snark G, and replacing each edge incident to v_1 or v_2 by semiedges.
ii. $(1, 1)$-semi-graph L' (superedge) is an isolated edge (two semiedges);
iii. $(1, 3, 3)$-semi-graph J' (supervertex) consists of two isolated edges and a vertex;

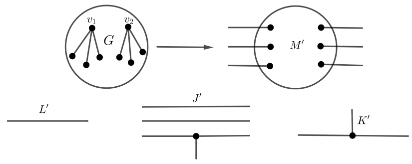

Figure 3. Superedge M' obtained from snark G, superedge L', supervertex J', and supervertex K' for Kochol superposition.

iv. $(1, 1, 1)$-semi-graph K' (supervertex) consists of a vertex and three semiedges.

Let $G' = (V', E')$ be a snark. Replace every edge $e \in E'$ by a superedge ξ, and every vertex $v \in V'$ by a supervertex ϑ. If $v \in V'$ is incident to $e \in E'$ then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges. The obtained cubic graph $G(\nu,\vartheta)$ is called superposition of G' with G and it is a snark [8]. Finally, we establish our main result, Theorem 3. First, we present a construction of an infinite family of snarks obtained from a Kochol superposition of Goldberg snarks with t-Blowup snarks, and show that all members of the presented family are Type 1.

For each odd $i \geq 3$, let $R'_t(i)$ be the snark obtained by a superposition of Goldberg snark G_i with a t-Blowup snark, $t \geq 6$. Let ξ' be the superedges formed by deleting, from the t-Blowup snark, with $t \geq 6$, the two nonadjacent vertices of semi-graph B'_4 depicted in Figure 2(b), i.e., superedges ξ'_1, ξ'_2, ξ'_3, ξ'_4, and ξ'_5, are obtained by deleting the vertices $\{c_2, d_1\}$, $\{e_1, e_2\}$, $\{e_1, d_2\}$, $\{c_2, f_3\}$, and $\{e_3, f_4\}$, respectively. To obtain $R'_t(3)$, we replace the edges u_1u_2, u_2u_3, and u_3u_1 by the corresponding superedges ξ'_1, ξ'_2 and ξ'_3, respectively. Second, note that by construction of these colored superedges we can make the junction of the remaining semiedges, which are those of the same color (colors assigned to u_{i+1} and $u_{i+1}v_{i+1}$, respectively in the coloring of G_3) and extreme vertices of different colors (obtained by removing non-adjacent vertices with colors same to $u_{i+1}u_{i+2}$ and u_iu_{i+1}, for each $1 \leq i \leq 3$, respectively, where $u_{3+1} = u_1$) between pairs of consecutive superedges accordingly, say ξ'_1 with ξ'_2, ξ'_2 with ξ'_3, and ξ'_3 with ξ'_1.

Similarly, the superpositions $R'_t(i)$ with odd $i \geq 5$ and $t \geq 6$ are obtained by replacing: the edges u_1u_2, u_2u_3, u_3u_4, and u_4u_5 by the superedges ξ'_1, ξ'_2, ξ'_3, and ξ'_4 respectively; the edges u_ju_{j+1}, for $5 \leq j \leq i$ and odd $i \geq 7$ ($u_{i+1} = u_1$), by the superedge ξ'_5, for odd j; and by the superedge ξ'_4 for even j; and make the junctions of the remaining semiedges between pairs of consecutive superedges ξ'_j and ξ'_{j+1} (colors assigned to u_{j+1} and $u_{j+1}v_{j+1}$, respectively in the coloring of G_i, for $3 \leq j \leq i$ and $i \geq 7$ ($u_{i+1} = u_1$), respectively. Again, following Kochol's construction, the remaining edges (resp. vertices) are replaced by superedges L' (resp. supervertices K') which is equivalent to maintain the original edges (resp. vertices) of G_i for i odd and $i \geq 5$. Finally, if $v \in V$ is incident to $e \in E$, then a connector in ϑ_v is linked with a connector in ξ_e through the junction of semiedges.

Our main result is proved by induction using the recursive construction of both Goldberg snark G_i, odd $i \geq 3$, with a t-Blowup snark.

Theorem 3. *All snarks $R'_t(i)$ obtained by a superposition of a Goldberg snark G_i, odd $i \geq 3$, with a t-Blowup snark, $t \geq 6$, have $\chi''(R'_t(i)) = 4$.*

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