Graceful chromatic number of the first Blanuša snarks

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Abstract. *A graceful k-coloring of a graph G is a proper vertex coloring $f\colon V(G) \to \{1, 2, \dots, k\}$, $k \geq 1$, which induces a proper edge coloring $f'\colon E(G) \to \{1, 2, \dots, k-1\}$ defined by $f'(uv) = |f(u) - f(v)|$, where $u, v \in V(G)$. Graceful colorings were introduced by Gary Chartrand, around 2015, as a variation of the well known graceful labeling introduced by Alexander Rosa around 1967. Numerous papers have been published on both subjects, and several challenging problems remain open. In this work, we investigate graceful colorings within the context of cubic graph classes such as Generalized Blanuša snarks B_i^1, for $i \geq 1$, and determine that the smallest positive integer k for which B_i^1 has a graceful k-coloring is 6.*

Resumo. *Uma k-coloração graciosa de um grafo G consiste em uma coloração própria de vértices $f\colon V(G) \to \{1, 2, \dots, k\}$, $k \geq 1$, que induz uma coloração própria de arestas $f'\colon E(G) \to \{1, 2, \dots, k-1\}$ definida por $f'(uv) = |f(u) - f(v)|$, onde $u, v \in V(G)$. Colorações graciosas foram introduzidas por Gary Chartrand, por volta de 2015, como uma variação da conhecida rotulação graciosa introduzida por Alexander Rosa por volta de 1967. Numerosos artigos foram publicados sobre ambos os temas e vários problemas desafiadores permanecem em aberto. Neste trabalho, investigamos colorações graciosas no contexto de classes de grafos cúbicos como Snarks de Blanuša Generalizados B_i^1, para $i \geq 1$, e determinamos que o menor inteiro positivo k para o qual B_i^1 possui uma k-coloração graciosa é 6.*

1. Introduction

Graph coloring has emerged as a valuable tool for modeling seemingly innocent, but challenging problems. Throughout the years, this approach contributed to the development of solutions, utilizing various graph coloring techniques as fundamental tools.

A *proper vertex coloring* of a graph G, where colors are assigned to the vertices of a graph G such that adjacent vertices have different colors, was the mathematical model employed by [Appel and Haken 1976] in 1976, to successfully solve the famous Four-Color Conjecture using computational methods. This conjecture was proposed in 1850 by F. Guthrie, a student of Augustus De Morgan, and establishes that only four colors are necessary to color a map, such that, regions (represented by the vertices of graph G) sharing a common boundary (represented by an edge between the corresponding vertices) are assigned different colors.

A *proper edge coloring* of a graph G, where colors are assigned to the edges of a graph G such that adjacent edges have different colors, is another mathematical tool employed by [Tait 1880] to find a reformulation for the Four-Color Conjecture. He established the equivalence between the Four-Color Conjecture and the statement that every planar bridgeless cubic graph has a chromatic index 3 (where the chromatic index of a graph is the minimum number of colors needed to obtain a proper edge coloring of the graph). This implied that if the Four-Color Conjecture were false, then a counterexample would be a planar bridgeless cubic graph with chromatic index 4.

Motivated by the finding of counterexamples of cubic graphs with chromatic index 4, [Gardner 1976] introduced the term *snarks* for nontrivial connected bridgeless cubic graphs that do not admit a proper edge coloring with 3 colors. The Petersen graph is recognized as the first example of a snark and the smallest such graph.

In 1967, Rosa introduced the well known graceful labelings, that extend, in a certain way, the idea of simultaneously properly coloring vertices and edges of a graph. A *graceful labeling* of a graph G is an injective function $g\colon V(G) \to \{0, 1, \ldots, |E(G)|\}$, that assigns the label $|g(u) - g(v)|$ to each edge uv of G, ensuring pairwise distinct edge labels. A dynamic survey on graceful and other related labelings, annually maintained by [Gallian 2022], contains more than 3500 cited papers, reflecting the significant interest in this research area.

Around 2015, Gary Chartrand proposed a variant of graceful labelings, which was initially investigated by [Bi et al. 2017]. A *graceful k-coloring* of a graph G is a proper vertex coloring f of G that assigns colors from the set $\{1, 2, \ldots, k\}$ to the vertices of G, and that induces a proper edge coloring of G, where the color for each edge $uv \in E(G)$ is $|f(u) - f(v)|$. The minimum positive integer k for which G has a graceful k-coloring is called the *graceful chromatic number* of G, and is denoted by $\chi_g(G)$.

In this work, we determine that each graph belonging to the first family of generalized Blanuša snarks has graceful chromatic number equal to 6.

2. First family of Generalized Blanuša Snarks

In 1946, [Blanuša 1946] introduced the *first Blanuša snark* (see Figure 1). Inspired by this graph, [Watkins 1983] constructed an infinite family of snarks that generalizes the first Blanuša snark, named *the first family of generalized Blanuša snarks*.

Let $\mathcal{B}^1 = \{B_1^1, B_2^1, B_3^1, \ldots\}$ be the first family of Generalized Blanuša Snarks. A snark $B_i^1 \in \mathcal{B}^1$, with $i \geq 1$, is formed by the union of block B_0^1 of Figure 2 with i copies of block B of Figure 3, called $B_1, B_2, \ldots, B_i$. We define the set of *link edges* as $E_{i-1,i} = \{c_{i-1}f_i, h_{i-1}a_i\}$, that are edges that link blocks B_{i-1} and B_i, $i \geq 2$ (see Figure 4).

Next, we define a recursive construction of the members of $\mathcal{B}^1$ that is used in the proof of our main result. The first graph $B_1^1 \in \mathcal{B}^1$, is defined as the union of B_0^1 and B_1 and the graph spanned by the edge set $\{tf_1, za_1, qh_1, wc_1\}$ (see Figure 1). The second graph $B_2^1 \in \mathcal{B}^1$, is defined as the union of B_0^1, B_1, B_2 and the graph spanned by the edge set $E_{1,2} \cup \{tf_1, za_1, qh_2, wc_2\}$ (see Figure 5). We define the *link-graph* L_i as the union of blocks B_{i-1}, B_i and the graph spanned by the link edges $E_{i-1,i}$ (see Figure 4). For each $i \geq 3$, B_i^1 is recursively obtained from the snark B_{i-2}^1 and the link-graph L_i, as follows:

- $V(B_i^1) = V(B_{i-2}^1) \cup V(L_i)$; and
- $E(B_i^1) = (E(B_{i-2}^1) \setminus E_{i-2}^{out}) \cup E(L_i) \cup E_i^{in}$, where $E_{i-2}^{out} = \{qh_{i-2}, wc_{i-2}\}$ and $E_i^{in} = E_{i-2,i-1} \cup \{qh_i, wc_i\}$.

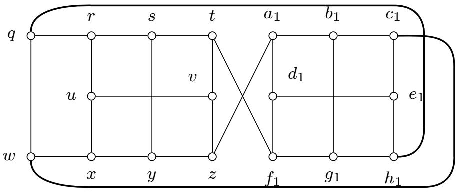


Figure 1. Snark B_1^1.

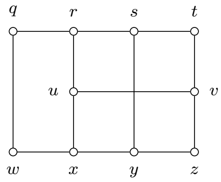


Figure 2. Block B_0^1.

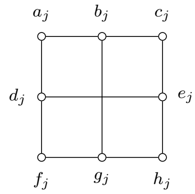


Figure 3. Block B.

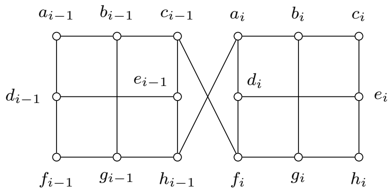
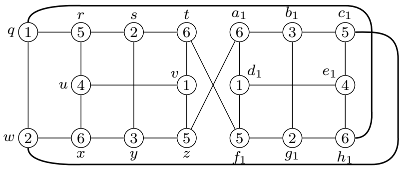


Figure 4. Link-graph L_i.

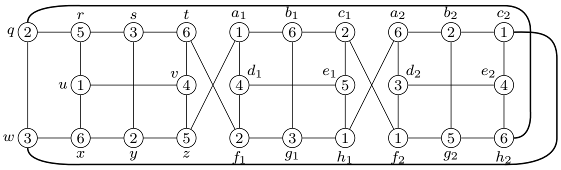
In the proof of the next theorem, we use the following result:

Lemma 1. *[Pantoja et al. 2024] If a cubic graph G contains the cycle C_5 as a subgraph, then $\chi_g(G) \geq 6$.*

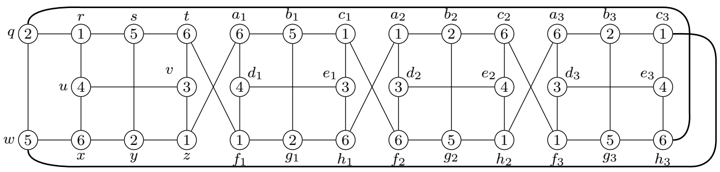
Next, we define the minimum positive integer k for which the members of the first family of generalized Blanuša snarks have a graceful k-coloring.



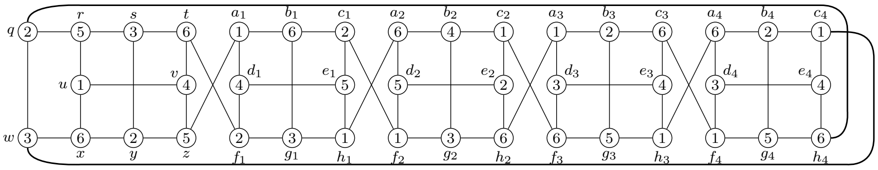
Generalized Blanuša Snark B_1^1.



Generalized Blanuša Snark B_2^1.



Generalized Blanuša Snark B_3^1.



Generalized Blanuša Snark B_4^1.

Figure 5. Graceful 6-colorings for B_1^1, B_2^1, B_3^1 and B_4^1.

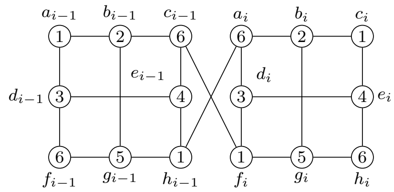


Figure 6. Graceful 6-coloring ϕ of link-graph L_i.

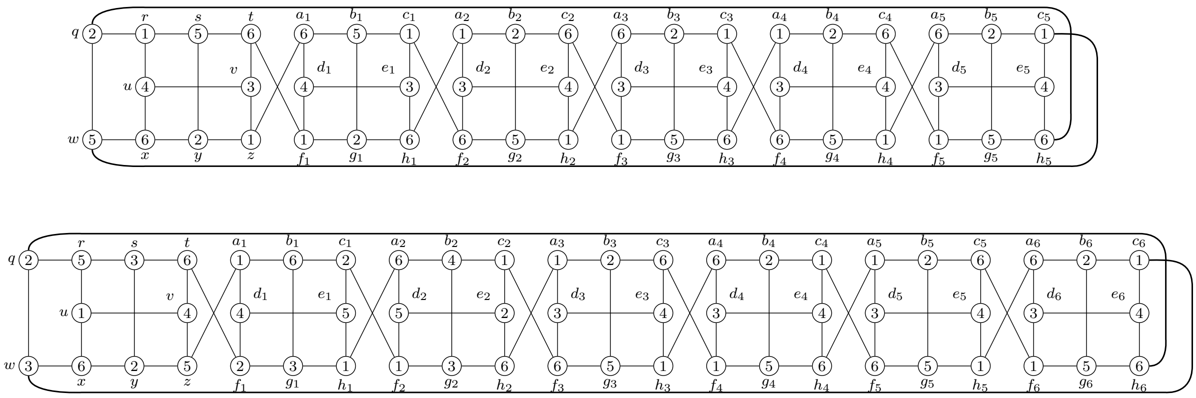


Figure 7. Graceful 6-colorings for snarks B_5^1 and B_6^1.

Theorem 2. *If B_i^1 is a generalized Blanuša snark, then $\chi_g(B_i^1) = 6$, for $i \geq 1$.*

Sketch of the proof. By Lemma 1, since C_5 is a induced subgraph of B_i^1, we have that $\chi_g(B_i^1) \geq 6$, for $i \geq 1$. To prove that $\chi_g(B_i^1) \leq 6$, for $i \geq 1$, we proceed by induction on i. We refer to Figure 5 for a graceful 6-coloring for B_1^1, B_2^1, B_3^1 and B_4^1.

We construct a graceful 6-coloring f_i for B_i^1, $i \geq 5$, from the graceful 6-coloring f_{i-2} of B_{i-2}^1 and the graceful 6-coloring ϕ (presented in Figure 6) of the link-graph L_i, as follows. For each vertex $v \in V(B_i^1)$, we define

$$f_i(v) = \begin{cases} f_{i-2}(v) & \text{if } v \in V(B_{i-2}^1) \cap V(B_i^1); \\ \phi(v) & \text{if } v \in V(L_i) \cap V(B_i^1). \end{cases}$$

We refer to Figure 7, for an example of a graceful 6-coloring f_5 for B_5^1 (resp. f_6 for B_6^1) obtained from the graceful 6-coloring f_3 of B_3^1 (resp. f_4 of B_4^1) and a graceful 6-coloring ϕ of the link-graph L_i presented in Figure 6. First, note that, the graceful 6-coloring of the induced subgraph $B_i^1\Big[B_0^1 \cup \Big(\bigcup\limits_{j=1}^{i-2} V(B_j)\Big)\Big]$ is a graceful 6-coloring because f_{i-2} is a graceful 6-coloring for B_{i-2}^1. The same applies to the graceful 6-coloring of $B_i^1\Big[V(B_{i-1}) \cup V(B_i)\Big]$, obtained from the graceful 6-coloring ϕ of Figure 6. We complete the proof by showing that f_i, restricted to the induced subgraph $B_i^1\Big[\bigcup_{j=i-2}^{i} V(B_j) \cup B_0^1\Big]$, is a graceful 6-coloring. $\square$

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