The line graphs of Möbius ladder graphs are Type 1

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Abstract. *A k-total coloring of G is an assignment of k colors to its elements (vertices and edges) such that adjacent or incident elements have distinct colors. The total chromatic number of a graph G is the smallest integer k for which G has a k-total coloring. If the total chromatic number of G is $\Delta(G) + 1$, then we say that G is Type 1. The line graph of G, denoted by $L(G)$, is the graph whose vertex set is the edge set of G and two vertices of the line graph of G are adjacent if the corresponding edges are adjacent in G. In this paper, we prove that the line graphs of Möbius ladder graphs, $L(M_{2n})$, are Type 1.*

1. Introduction

Let $G = (V, E)$ be a simple connected graph. A *k-vertex coloring of G* is an assignment of k colors to the vertices of G so that adjacent vertices have different colors. A *k-edge coloring of G* is an assignment of k colors to the edges of G so that adjacent edges have different colors. The *chromatic index of* G, denoted by $\chi'(G)$, is the smallest k for which G has a k-edge coloring. Vizing's theorem states that the chromatic index $\chi'(G)$ is at least $\Delta(G)$ and at most $\Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the graph G [Vizing 1964]. Graphs with $\chi'(G) = \Delta(G)$ are called *Class 1*, and graphs with $\chi'(G) = \Delta(G) + 1$ are called *Class 2*. A *k-total coloring of G* is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number of* G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$ and the Total Coloring Conjecture (TCC) states that the total chromatic number of any graph is at most $\Delta(G) + 2$ [Behzad 1965, Vizing 1964]. Graphs with $\chi''(G) = \Delta(G) + 1$ are called *Type 1*, and graphs with $\chi''(G) = \Delta(G) + 2$ are called *Type 2*.

A vertex coloring $\varphi : V(G) \to \{1, 2, \ldots, \Delta(G) + 1\}$ is called *conformable* if the number of color classes (including empty color classes) of parity different from that of $|V(G)|$ is at most $def(G) = \sum_{v \in V(G)}(\Delta(G) - d(v))$. Note that if G is a regular graph, then φ is called conformable if each color class has the same parity as $|V(G)|$. A graph is said to be *conformable* if it has a conformable vertex coloring; otherwise, it is said to be *non-conformable*. Let $n \geq 3$ be a positive integer. The *Möbius ladder* M_{2n} is the graph with vertex set $V(M_{2n}) = \{u_i, v_i \mid i \in \{0, \ldots, n-1\}\}$ and edge set $E(M_{2n}) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i \mid i \in \{0, 1, \ldots, n-2\}\} \cup \{u_{n-1}v_{n-1}, u_{n-1}v_0, v_{n-1}u_0\}$. The *line graph* of G, denoted by $L(G)$, is the graph whose vertex set is the edge set of G, and two vertices of $L(G)$ are adjacent if the corresponding edges are adjacent in G. We set the vertices of $L(M_{2n})$ as $u'_i = u_iu_{i+1}$, $v'_i = v_iv_{i+1}$ for $i \in \{0, \ldots, n-2\}$; $m_i = u_iv_i$ for $i \in \{0, \ldots, n-1\}$; $u'_{n-1} = u_{n-1}v_0$; and $v'_{n-1} = v_{n-1}u_0$.

An important connection between total chromatic number and conformability of graphs was established in Theorem 1.

Theorem 1 ([Chetwynd and Hilton 1988])**.** *If G is non-conformable, then G is not Type 1.*

It is known that determining the chromatic index and total chromatic number are NP-complete problems even for regular graphs [Leven and Galil 1983, McDiarmid and Sánchez-Arroyo 1994]. [Vignesh et al. 2018] conjectured that all line graphs of complete graphs $L(K_n)$ are *Type 1*. [Mohan et al. 2021] verified the TCC to the set of quasi-line graphs, which is a generalization of line graphs, and present some infinite families of *Type 1* graphs. [Jayaraman et al. 2022] determined the total chromatic number for certain line graphs. [Faria et al. 2023] determined the conformability of line graphs $L(G)$ when G is *Class 1* (Theorem 2), and proposed Question 1.

Theorem 2 ([Faria et al. 2023])**.** *Let G be a k-regular graph. If G is Class 1, then $L(G)$ is conformable.*

Question 1 ([Faria et al. 2023])**.** *Is there a k-regular graph G, $k \geq 3$, such that the line graph $L(G)$ is non-conformable?*

[Chetwynd and Hilton 1988] proved that M_{2n} is *Type 2*, for all $n \geq 3$. From Theorem 1, non-conformable graphs are *Type 2*. Since M_{2n} is *Class 1*, $L(M_{2n})$ is conformable. In order to investigate Question 1, we extend the search for *Type 2* line graphs from regular graphs which are conformable. Furthermore, we propose Conjecture 1.

Conjecture 1. *If G is a k-regular Class 1 graph, then $L(G)$ is Type 1.*

Next, we prove that $L(M_{2n})$ is *Type 1*, for $n \geq 3$, which supports Conjecture 1.

2. Main Result

We start this section by presenting a recursive process to construct $L(M_{2n})$. Let $n \geq 3$ and a positive integer $i \in \{0, 1, \ldots, n-2\}$. Observe that $L(M_{2(n+1)})$ can be obtained from $L(M_{2n})$ by the recursive relation:

$$V(L(M_{2(n+1)}) := (V(L(M_{2n})) \setminus \{u'_{i+1}, v'_{i+1}\}) \cup \{u'_\ell, u'_r, v'_\ell, v_r, m\} \tag{1}$$

$$E(L(M_{2(n+1)}) := E(L(M_{2n})) \cup A \tag{2}$$

where $A = \{u'_i u_\ell, u'_\ell u'_r, u'_r u'_{i+2}, v'_i v'_\ell, v'_\ell v'_r, v'_r v'_{i+2}, u'_\ell m, u'_r m, v_\ell m, v'_r m\}$. The recursive relation is presented on the induced subgraph of $L(M_{2n})$ by set $S_i = \{u'_i, u'_{i+1}, u'_{i+2}, m_{i+1}, m_{i+2}, v'_i, v'_{i+1}, v'_{i+2}\}$ in Figure 1. The operation consisting of removing vertices u'_{i+1} and v'_{i+1}, adding vertices $u'_\ell, u'_r, v'_\ell, v_r$ and m, and adding edges in A is called an *extension of the Möbius line* to $L(M_{2n})$. Note that the application of this recursive operation generates the subsequent members of the family $L(M_{2n})$. When this operation is applied k times, it is called a *k-extension of the Möbius line* to $L(M_{2n})$.

Theorem 3. *The graph $L(M_{2n})$ is Type 1.*

Proof. Let $n \geq 3$. We present 5-total colorings for all graphs $L(M_{2n})$, note that these graphs are 4-regular, and so all these graphs are *Type 1*. In this proof, we consider two cases on the parity of n. In both cases, the 5-total colorings of the subsequent members are obtained by preserving the colors of the preserved elements of colored graphs $L(M_6)$ and $L(M_8)$ (one for each case), and we complete the 5-total colorings of the subsequent members, obtaining 5-total colorings for the induced subgraph involved in the recursive operation of the family.

1) Suppose that $n \geq 3$ is odd. Consider a 5-total coloring ϕ_1 to $L(M_6)$ presented in Figure 2a. In order to obtain any graph in this case, we observe that for $k \geq 1$,

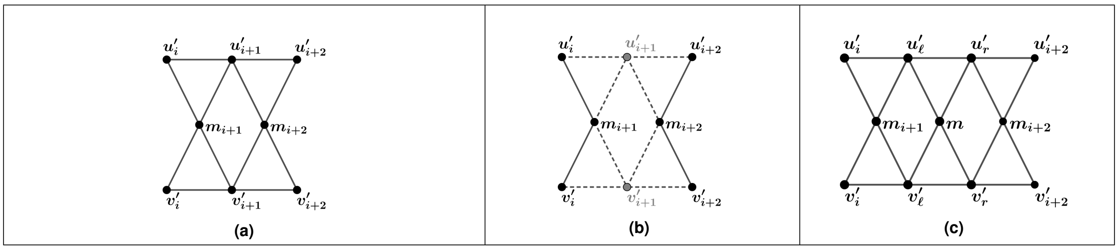

Figure 1. A depiction of the recursive operation to obtain $L(M_{2(n+1)})$ from $L(M_{2n})$. In 1a, the induced subgraph of $L(M_{2n})$ by the set $S_i = \{u'_i, u'_{i+1}, u'_{i+2}, m_{i+1}, m_{i+2}, v'_i, v'_{i+1}, v'_{i+2}\}$. In 1b, the gray vertices u'_{i+1} and v'_{i+1} means that both vertices are removed from the graph. In 1c, the recursive process ends by adding vertices $u'_\ell, u_r, v'_\ell, v'_r, m$ and edges of A.

the application of the $2k$-extension of the Möbius line to $L(M_6)$ generates the graph $L(M_{6+4k}) \simeq L(M_{2(2k+3)})$ such that $n = 2k+3$. Let $G[S]$ be the induced subgraph of $L(M_{2(2k+3)})$ by the set $S = \{u'_j, m_j, v'_j \mid j \in \{1, 2, \ldots, 2k+1\}\}$. Hence, we obtain a 5-total coloring ϕ'_1 to $L(M_{2(2k+3)})$ in the following way:

Suppose that $x \in V(L(M_{2(2k+3)}))$. If $x \notin V(G[S])$, then x is an element of the set $X_1 = \{u'_0, u'_{2k+2}, m_0, m_1, m_{2k+2}, v'_0, v'_{2k+2}\}$. In this case, we assign $\phi'_1(x) = \phi_1(x)$, where u'_{2k+2}, m_{2k+2} and v'_{2k+2} of $V(L(M_{2(2k+3)}))$ are associated to u'_2, m_2 and v'_2 of $V(L(M_6))$, respectively (see Figure 2c). If $x \in V(G([S])$, then we assign for $j \in \{1, 2, \ldots, 2k+1\}$, $\phi'_1(u'_j) = 1$ if j is odd or $\phi'_1(u'_j) = 5$ if j is even; $\phi'_1(v'_j) = \phi'_1(u'_j)$ and; $\phi'_1(m_j) = 4$ (see Figure 2d).

Suppose that $x \in E(L(M_{2(2k+3)}))$. If $x \notin E(G[S])$, then x is an incident edge of an vertex in X_1. In this case, we assign $\phi'_1(x) = \phi_1(x)$, where similarly u'_{2k+2}, m_{2k+2} and v'_{2k+2} of $V(L(M_{2(2k+3)}))$ are associated to u'_2, m_2 and v'_2 of $V(L(M_6))$, respectively (see Figure 2c). If $x \in E(G[S])$, then we assign for $j \in \{1, 2, \ldots, 2k\}$, $\phi'_1(u'_j u'_{j+1}) = 4$ if j is odd or $\phi'_1(u'_j u'_{j+1}) = 2$ if j is even; $\phi'_1(v'_j v'_{j+1}) = 4$ if j is odd or $\phi'_1(v'_j v'_{j+1}) = 3$ if j is even; $\phi'_1(u'_j m_{j+1}) = 5$ if j is odd or $\phi'_1(u'_j m_{j+1}) = 1$ if j is even; $\phi'_1(u'_{j+1} m_{j+1}) = 3$; $\phi'_1(v'_{j+1} m_{j+1}) = 1$ if j is odd or $\phi'_1(v'_{j+1} m_{j+1}) = 5$ if j is even and; $\phi'_1(v'_j m_{j+1}) = 2$ (see Figure 2d).

It is straightforward that ϕ'_1 is a 5-total coloring of $L(M_{2(2k+3)})$. Figures 2d presents the 5-total coloring ϕ'_1 restricted to vertices and edges of $G[S]$.

2) Suppose that $n \geq 4$ is even. Consider a 5-total coloring ϕ_2 to $L(M_8)$ presented in Figure 2b. In order to obtain any graph in this case, we observe that for $k \geq 1$, the application of $2k$-extension of the Möbius line to $L(M_8)$ generates the graph $L(M_{8+4k}) \simeq L(M_{2(2k+4)})$ such that $n = 2k+4$. Let $G[S]$ be the induced subgraph of $L(M_{2(2k+4)})$ by the set $S = \{u'_j, m_j, v'_j \mid j \in \{1, 2, \ldots, 2k+1\}\}$. Hence, we obtain a 5-total coloring ϕ'_2 to $L(M_{2(2k+4)})$ in the following way:

Suppose that $x \in V(L(M_{2(2k+4)}))$. If $x \notin V(G[S])$, then x is an element of the set $X_2 = \{u'_0, u'_{2k+2}, u'_{2k+3}, m_0, m_1, m_{2k+2}, m_{2k+3}, v'_0, v'_{2k+2}, v'_{2k+3}\}$. In this case, we assign $\phi'_2(x) = \phi_2(x)$, where u'_{2k+2}, u'_{2k+3}, m_{2k+2}, m_{2k+3}, v'_{2k+2} and v'_{2k+3} of $V(L(M_{2(2k+4)}))$ are associated to u'_2, u'_3, m_2, m_3 and v'_2, v'_3 of $V(L(M_8))$, respectively (see Figure 2e). If $x \in V(G([S])$, then we assign for $j \in \{1, 2, \ldots, 2k+1\}$, $\phi'_2(u'_j) = 2$ if j is odd or $\phi'_2(u'_j) = 3$ if j is even; $\phi'_2(v'_j) = \phi'_2(u'_j)$ and; $\phi'_2(m_j) = 1$ (see Figure 2f).

Suppose that $x \in E(L(M_{2(2k+4)}))$. If $x \notin E(G[S])$, then x is an incident edge of an vertex in X_2. In this case, we assign $\phi_2'(x) = \phi_2(x)$, where similarly u'_{2k+2}, u'_{2k+3}, m_{2k+2}, m_{2k+3}, v'_{2k+2} and v'_{2k+3} of $V(L(M_{2(2k+3)}))$ are associated to u'_2, u'_3, m_2, m_3, v'_2 and v'_3 of $V(L(M_8))$, respectively (see Figure 2e). If $x \in E(G[S])$, then we assign for $j \in \{1, 2, \ldots, 2k\}$, $\phi_2'(u'_j u'_{j+1}) = 5$ if j is odd or $\phi_2'(u'_j u'_{j+1}) = 1$ if j is even; $\phi_2'(v'_j v'_{j+1}) = 4$ if j is odd or $\phi_2'(v'_j v'_{j+1}) = 1$ if j is even; $\phi_2'(u'_j m_{j+1}) = 3$ if j is odd or $\phi_2'(u'_j m_{j+1}) = 2$ if j is even; $\phi_2'(u'_{j+1} m_{j+1}) = 4$ $\phi_2'(v'_{j+1} m_{j+1}) = 2$ if j is odd or $\phi_2'(v'_{j+1} m_{j+1}) = 3$ if j is even and; $\phi_2'(v'_j m_{j+1}) = 5$ (see Figure 2f).

It is straightforward that ϕ_2' is a 5-total coloring of $L(M_{2(2k+3)})$. Figure 2f presents the 5-total coloring ϕ_2' restricted to vertices and edges of $G[S]$. □

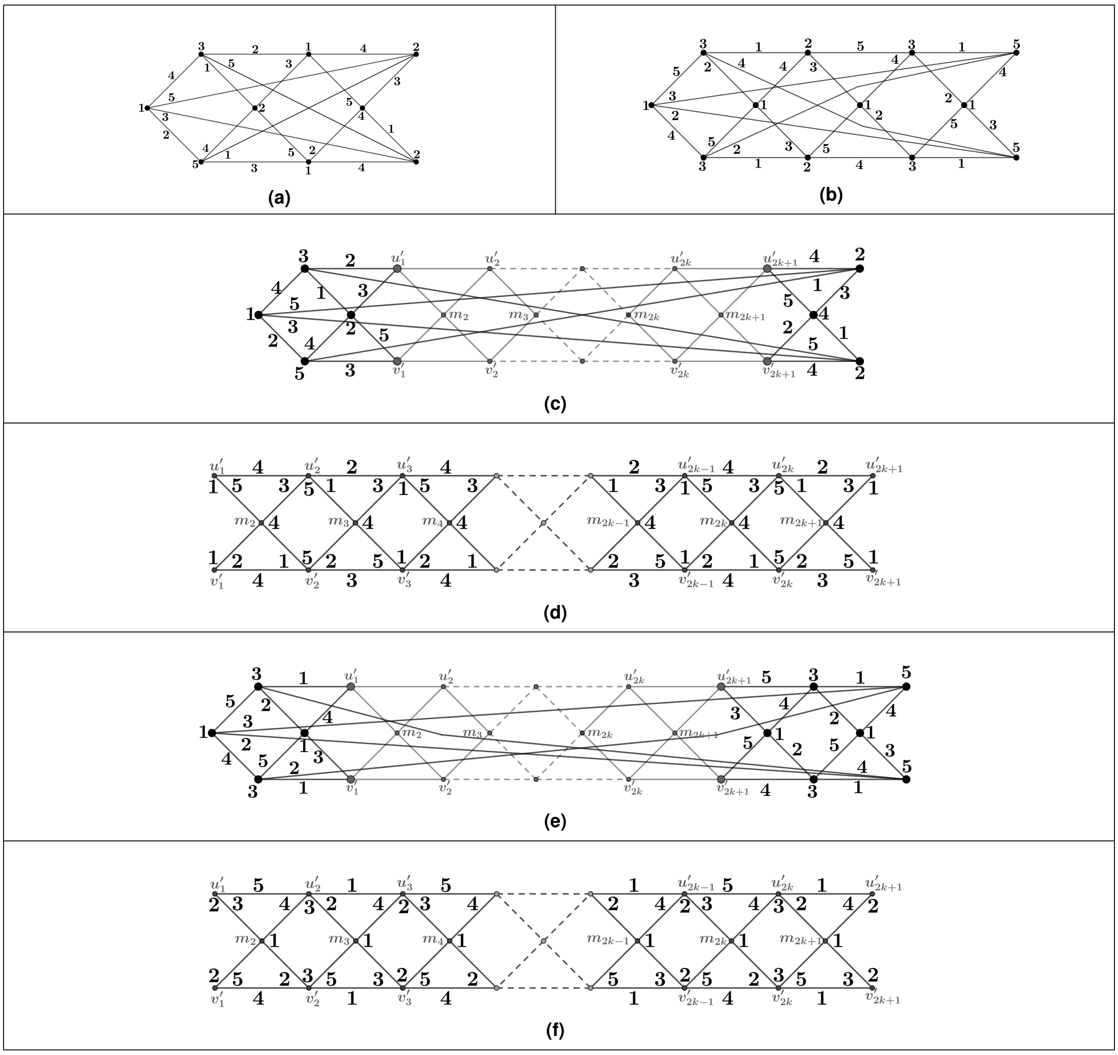

Figure 2. A depiction of the construction of the 5-total coloring of $L(M_{2n})$. In 2a and 2b, 5-total colorings ϕ_1 of $L(M_6)$ and ϕ_2 of $L(M_8)$, respectively. In 2c for $n \geq 3$ odd (resp. 2e for $n \geq 4$ even), the $2k$-extension of $L(M_6)$ (resp. $L(M_8)$) which obtains $L(M_{2(2k+3)})$ (resp. $L(M_{2(2k+4)})$. The elements of $L(M_6)$ (resp. $L(M_8)$) are preserved in all subsequent members, and they receive the colors assigned by ϕ_1 (resp. ϕ_2). In 2d (resp. 2f), the elements of $G[S]$ colored by ϕ_1' (resp. ϕ_2'). Together with 2c (resp. 2e), a 5-total coloring ϕ_1' (resp. ϕ_2') of $L(M_{2(2k+3)})$ (resp. $L(M_{2(2k+4)})$ is obtained.

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