Clique-Number of Timbral Graphs

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Abstract. *We study the clique-number of the timbral graphs $T_{n,k,\ell}$. The vertex set of $T_{n,k,\ell}$ is the set of all words of length k built on an alphabet of n symbols and two vertices are adjacent when they agree in exactly ℓ coordinates. We provide lower and upper bounds for the general case and determine $\omega(T_{n,k,1})$ when $k-1 \leq n$ is a prime power, showing the correspondence between a clique with n^2 vertices in $T_{n,n+1,1}$ and an affine plane of order n.*

1. Introduction

We study the clique-number of the timbral graphs $T_{n,k,\ell}$, first mentioned in [Akhmedov and Winter 2014]. We provide lower and upper bounds for the general case and determine $\omega(T_{n,k,1})$ when $k-1 \leq n$ is a prime power, showing the correspondence between a clique with n^2 vertices in $T_{n,n+1,1}$ and an affine plane of order n.

The vertex set of $T_{n,k,\ell}$ is the set of all words of length k built on an alphabet of n symbols and two vertices are adjacent when they agree in exactly ℓ coordinates. In [Akhmedov and Winter 2014], the authors define the timbral graph as a model for a very specific way of composing music, based on the enumeration of musical elements through hamiltonian cycles.

This graph class generalizes the Hamming graphs, which are defined in the same set of vertices as $T_{n,k,\ell}$, being that two vertices are adjacent when they differ by exactly one symbol, that is, when they agree in $k-1$ coordinates.

In the bibliography on graph theory, we find many other classes of graphs which are related to or generalize the Hamming graphs [Sharifiyazdi 2007, Imrich and Klavzar 1996, Harney 2017]. One of those, which has many applications in coding theory [Lint 1995], is the class of the Hamming-distance graphs $H_n(k,d)$ [Harney 2017], in which two vertices are adjacent when they differ in *at least* d coordinates. In some papers these graphs are ambiguously called Hamming graphs [Rouayheb and Georghiades 2011, Sloane 1989].

One of the main connections between $H_n(k,d)$ and the coding theory, as posed in [Sloane 1989], is the problem of finding the maximum size $A_n(k,d)$ of a code whose words are elements of $\mathbb{Z}_n^k = \underbrace{\mathbb{Z}_n \times \cdots \times \mathbb{Z}_n}_{k \text{ times}}$ and the minimum hamming distance between any two words is greater than or equals to d. In graph theory, this problem is equivalent to finding $\omega(H_n(k,d))$, the maximum size of a clique in the Hamming-distance graph. We also observe that the graph $H_n(k,d)$ is the union of the graphs $T_{n,k,\ell}$, when ℓ varies from 0 to $k-d$. Then, the study of $\omega(T_{n,k,\ell})$ may shed some light on $A_n(k,d)$. On the other hand, this close connection between the Hamming-distance and the timbral graphs can broaden the perspective from which $T_{n,k,\ell}$ can be studied.

In what follows, n, k and ℓ are nonnegative integers such that $n \geq 2$ and $k \geq \ell+1$.

2. Bounds on $\omega(T_{n,k,\ell})$

To obtain bounds for the clique number of timbral graphs, it is wise to first investigate the particular case when $\ell = 0$. Note that if we take a word $u \in \mathbb{Z}_n^\ell$, the subset of vertices of $T_{n,k,\ell}$ that are prefixed by u induces a subgraph isomorphic to $T_{n,k-\ell,0}$. Thus, $\omega(T_{n,k-\ell,0})$ is a lower bound for $\omega(T_{n,k,\ell})$. Our search for bounds on the clique number of timbral graphs begins with the following proposition.

Proposition 1. *$\omega(T_{n,k,0}) = n$.*

As we have seen, whenever $\ell \geq 1$, $T_{n,k,\ell}$ has a subgraph isomorphic to $T_{n,k-\ell,0}$. Thus, we have:

Corollary 1. *$\omega(T_{n,k,\ell}) \geq n$.*

The proof of Proposition 1 uses the fact that $T_{n,k,0}$ is a n-partite graph. To deal with the general case, we can adapt this idea to show that $T_{n,k,\ell}$ is a ($n^{\ell+1}$)-partite graph. This gives us the following:

Theorem 1. *$\omega(T_{n,k,\ell}) \leq n^{\ell+1}$.*

Before we proceed, we need some definitions.

Definition 1. *Let K be a clique of $T_{n,k,\ell}$, $u \in K$, and S be a subset of $\mathbb{Z}_k$ such that $|S| = \ell$. The (u,S)-*part *of K is the set $K_S(u) = \{v \in K : I_c(u,v) = S\}$, where $I_c(u,v) = \{i \in \mathbb{Z}_k : u_i = v_i\}$.*

We observe that $u \notin K_S(u)$, since $\ell < k$. Our goal is to obtain a new bound for $\omega(T_{n,k,\ell})$ by stipulating how many non-empty (u,S)-parts can coexist in a clique K and what is the maximum size of these parts.

Definition 2. *Let K be a clique of $T_{n,k,\ell}$ and $u \in K$. We denote by $\mathcal{P}_K(u)$ the family of the ℓ-subsets of $\mathbb{Z}_k$ such that $K_S(u)$ is non-empty. Formally, we have $\mathcal{P}_K(u) = \{S \in \binom{\mathbb{Z}_k}{\ell} : K_S(u) \neq \emptyset\}$.*

For a fixed vertex u, the set $\{K_S(u) : S \in \mathcal{P}_K(u)\}$ is a partition of $K \setminus \{u\}$.

Theorem 2. *Let K be a clique of $T_{n,k,\ell}$ and $u \in K$. Then, $|K| \leq |\mathcal{P}_K(u)|(n-1)+1$. As a consequence, $\omega(T_{n,k,\ell}) \leq \binom{k}{\ell}(n-1)+1$.*

Proof. Let K be a clique that contains u. For any $S \in \binom{\mathbb{Z}_k}{\ell}$, the set $K_S(u) \cup \{u\}$ corresponds to a clique of $T_{n,k-\ell,0}$, since we can distinguish the vertices in $K_S(u)$ only by the coordinates in $\mathbb{Z}_k \setminus S$ and any two vertices in $K_S(u)$ differ in all these coordinates. Thus, by Proposition 1, $|K_S(u) \cup \{u\}| = |K_S(u)| + 1 \leq \omega(T_{n,k-\ell,0}) = n$, which means that $|K_S(u)| \leq n-1$. Since we are partitioning K in $|\mathcal{P}_K(u)|$ distinct non-empty parts of the form $K_S(u)$ and a part $\{u\}$, the theorem holds. ■

In many cases, this bound is not tight, as we are not taking into account the relationship between vertices at different parts–although $\omega(T_{n,k,0})$ reaches the bound, as we have seen. If $|\mathcal{P}_K(u)| = 1$, then $n-1$ is the best bound we can achieve for the size of the part. However, if there are at least two distinct parts in $\mathcal{P}_K(u)$, we can obtain a new upper bound for the size of K.

Lemma 1. *Let K be a clique and $u \in K$. If $\{K_S(u), K_{S'}(u)\} \subseteq \mathcal{P}_K(u)$, then*

$$|K_S(u)| \leq \left\lfloor \frac{k - 2\ell + |S \cap S'|}{\ell - |S \cap S'|} \right\rfloor.$$

Since the bound in Lemma 1 increases with $|S \cap S'|$, the most restrictive upper bound for $K_S(u)$, for fixed S, is obtained through the subset $S' \in \mathcal{P}_K(u)$ that minimizes $|S \cap S'|$. Accordingly, we define $\mu(S) = \min\{|S \cap S'| : S' \in \mathcal{P}_K(u)\}$. Now, we apply the bound in Lemma 1 to obtain an upper bound for the size of K.

Theorem 3. *Let K be a clique and $u \in K$. If $|\mathcal{P}_K(u)| \geq 2$, then*

$$|K| \leq 1 + \sum_{S \in \mathcal{P}_K(u)} \left\lfloor \frac{k - 2\ell + \mu(S)}{\ell - \mu(S)} \right\rfloor.$$

As a consequence, $\omega(T_{n,k,\ell}) \leq \max\{1 + \binom{k}{\ell}(k - \ell - 1), n\}$.

The bound in Theorem 3 is not very helpful when we do not know much about the structure of the clique K, but it can be particularly useful in some cases, as of when $\ell = 1$, which is the subject of the next section.

3. Clique-Number of $T_{n,k,1}$

In this section, we consider the case where $\ell = 1$. Combining the bounds obtained in Section 2, we first reduce the problem to the analysis of three cases:

1. If $n < k - 1$, then Theorem 1 gives us our best upper bound, which states that $\omega(T_{n,k,1}) \leq n^2$.
2. If $k - 1 \leq n < (k-1)^2$, then $\omega(T_{n,k,1}) \leq (k-1)^2$ is our best upper bound, given by Theorem 3.
3. If $(k-1)^2 \leq n$, then $\omega(T_{n,k,1}) = n$, combining theorems 1 and 3.

As we have equality in Case 3, we are left with the problem of investigating the bounds obtained in the other two cases. From now on, we consider $k - 1 \leq n < (k-1)^2$, leaving the analysis of Case 1 for another occasion. Our main result is:

Theorem 4. *Let $k - 1 \leq n < (k-1)^2$. Then, $\omega(T_{n,k,1}) \geq (k-1)^2$ if and only if there is an affine plane of order $k - 1$.*

We recall that an *affine plane* is a structure $\mathbb{A} = (\mathfrak{P}, \mathfrak{L}, I)$, where $\mathfrak{P}$ is the set of *points*, $\mathfrak{L}$ is the set of *lines*, and $I \subseteq \mathfrak{P} \times \mathfrak{L}$ is the *incidence relation*, satisfying:

i. Every two points are incident to a unique common line.
ii. Given a line l and a point P that is not incident to l, there is a unique line l' that is incident to P and does not intersect l.
iii. There are four points such that no three of them are incident to the same line.

The *order* of an affine plane is the number of points incident to a single line, which happens to be the same for every line in $\mathfrak{L}$. Also, an affine plane of order n has n^2 points and $n(n+1)$ lines, which can be partitioned in $n+1$ classes of parallel lines [Shult 2010].

We break the proof of Theorem 4 into two lemmas. First, we show that every clique K of $T_{n,k,1}$ with $(k-1)^2$ vertices is such that only $k-1$ values are represented in each coordinate, that is, the set $R_i = \{v_i : v \in K\}$ has exactly $k-1$ elements, for all $i \in \mathbb{Z}_k$. Thus, we can build bijections σ_i between R_i and $\mathbb{Z}_{k-1}$ to get a correspondent clique in $T_{k-1,k,1}$.

Lemma 2. *Let $k-1 \leq n < (k-1)^2$. Then, $\omega(T_{n,k,1}) = (k-1)^2$ if and only if $\omega(T_{k-1,k,1}) = (k-1)^2$.*

Next, we show that a clique of $T_{n,n+1,1} = T_{k-1,k,1}$ with n^2 vertices can be viewed as a representation of an affine plane of order n, where the vertices are points and the edges represent the existence of a unique line between two points.

Lemma 3. *Let $k-1 \leq n < (k-1)^2$. Then, $\omega(T_{n,n+1,1}) \geq n^2$ if and only if there is an affine plane of order n.*

It is known that there is an affine plane of order n whenever n is a prime power, which comes from the finite field $\mathbb{F}_n$. Thus, we obtain:

Corollary 2. *If $n \geq k-1$ is a prime power, then $\omega(T_{n,k,1}) \geq (k-1)^2$.*

Moreover, for all $\ell \geq 1$, there is a subgraph of $T_{n,k,\ell}$ isomorphic to $T_{n,k-\ell+1,1}$, which gives us the following:

Corollary 3. *If $\ell \geq 1$ and $k-\ell \leq n$ is a prime power, then $\omega(T_{n,k,\ell}) \geq (k-\ell)^2$.*

4. Perspectives

We review our results for the clique-number of $T_{n,k,1}$, for a better understanding of what remains to be done. The cases we treated here were:

- If $n < k-1$, then $n \leq \omega(T_{n,k,1}) \leq n^2$.
- If $k-1 \leq n < (k-1)^2$, we have two cases:
  - If $k-1$ is a prime power, then $\omega(T_{n,k,1}) = (k-1)^2$.
  - If $k-1$ is not a prime power, then $n \leq \omega(T_{n,k,1}) \leq (k-1)^2$.
- If $(k-1)^2 \leq n$, then $\omega(T_{n,k,1}) = n$.

A natural question is that of knowing in which cases the upper bound given by Theorem 4 is tight when $k-1 \leq n < (k-1)^2$ and $k-1$ is not a prime power. Since the existence of an affine plane whose order is not a prime power is currently an open question, we do not know how to determine $\omega(T_{n,k,1})$ in this case.

We leave the task of studying the impact of our results on the problem of determining $A_n(k,d)$ for future work. There is still room, also, to improve our bounds on $\omega(T_{n,k,\ell})$, observing how the bounds apply to other values of ℓ. As we saw above, even the case when $\ell = 1$ is not yet closed, as we do not know the exact behavior of $\omega(T_{n,k,1})$ neither when $k-1$ is not a prime power nor when $n < k-1$. Moreover, the connection between a maximum clique of $T_{n,n+1,1}$ and an affine plane of order n may give rise to another approach to investigate the existence of affine planes of order n. Finally, one could return to musical composition with our results to see how they might affect or even inspire a compositional method based on timbral cliques.

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