Firefighters work better when the bandwidth is small*

Sérgio Fusquino[1*], **Luerbio Faria**[1], **Diana Sasaki**[1], **Vinicius dos Santos**[2]

[1]Institute of Mathematics and Statistics
Rio de Janeiro State University (UERJ) – Rio de Janeiro, RJ – Brazil

[2]Computer Science Department
Minas Gerais Federal University – Minas Gerais, MG – Brazil

sergio.fusquino@pos.ime.uerj.br, luerbio@ime.uerj.br,

diana.sasaki@ime.uerj.br, viniciussantos@dcc.ufmg.br

Abstract. *In this work we address the Firefighter problem in graphs and its relationship with the bandwidth parameter $bw(G)$ of a graph G. The Firefighter problem consists of a scenario in which a vertex v of the graph is initially set on fire, which we call the focus of the fire. The objective is to defend the largest number of vertices not on fire with firefighters, protecting vertex by vertex, as the fire spreads after each new defense. The bandwidth parameter in graphs is a minimum natural number, such that it is found after an optimal linear arrangement of the vertices, such that the distance between the indices of the vertices of this linear arrangement is the smallest possible. We relate this parameter to Firefighter to find a lower bound on the maximum number of vertices saved from fire in a graph G.*

1. Firefighter problem

In 1995, [Hartnell 1995] introduced the *Firefighter problem* in graphs, which consists of a fire starting at one vertex of a graph and then a non-burned vertex is chosen to be defended, making it unburnable. At each new step, the fire spreads to all adjacent vertices that were not defended in the previous steps and, again, one vertex can be defended by firefighters, until the fire stops spreading. We want to save as many vertices as possible, but this may not be very easy. Let $sn(G, v)$ denote the maximum number of vertices that can be saved when a fire breaks out at vertex v of graph G. An example can be found in Figure 1, with an optimal strategy for defending the graph. Throughout the examples in this paper, red vertices represent the burned vertices, and the subscript of label $b_k, k \geq 1$ indicates the step k at which the fire reached the respective vertex. Similarly, blue vertices represent the defended vertices which will be labeled by $d_k, k \geq 1$, and once again the subscript indicates the defense step. Black vertices in the end of the process are indirectly defended because the fire was contained.

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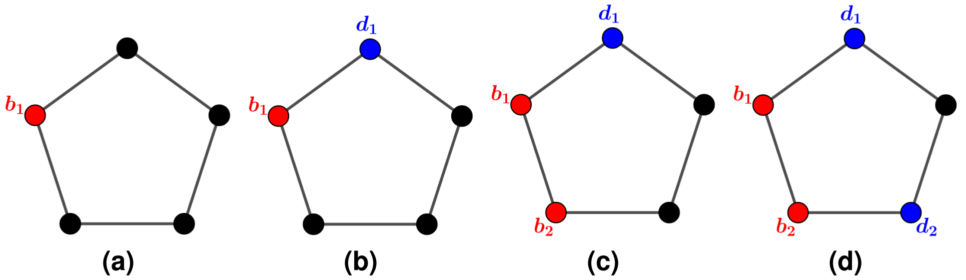

Figure 1. The 4 steps of Firefighter problem in graph C_5.

2. Bandwidth

A *linear layout* of G is a bijection $\pi : V \rightarrow \{1, \dots, n\}$. For convenience, we express π by the list $L = (v_1, \dots, v_n)$ where $\pi(v_i) = i$. Given a linear layout, we denote the distance between two vertices in L by $dist(v_i, v_j) = |j - i|$. The *bandwidth* $bw(G)$ of a graph G is the minimum $k \in \mathbb{N}$ such that the vertices of G can be arranged in a linear layout $L = (v_1, \dots, v_n)$ so that $dist(v_i, v_j) \leq k$ for every $v_i v_j$ of $E(G)$ [Chung 1988]. We want to find the labeling which minimizes the maximum "stretch" of all the edges, i.e. we want short edges in our linear layout. In Figure 2, we present a graph with its optimal linear layout and $bw(G) = 5$. Let P_n^k be the k^{th} power of the path P_n, in which two vertices $u, v \in V(P_n)$ are adjacent if, and only if, $0 < |dist(u, v)| \leq k$. A well-known result says that $bw(G) \leq k$ if and only if G is a subgraph of P_n^k [Chvátal 1970]. Note that the graph in Figure 2 is a subgraph of P_{21}^5 and there are edges whose endvertices have distance 5 in the optimal linear layout. So in this case, $bw(G) = 5$. In general, to find the bandwidth of a graph G is an NP-complete problem [Garey and Johnson 1979].

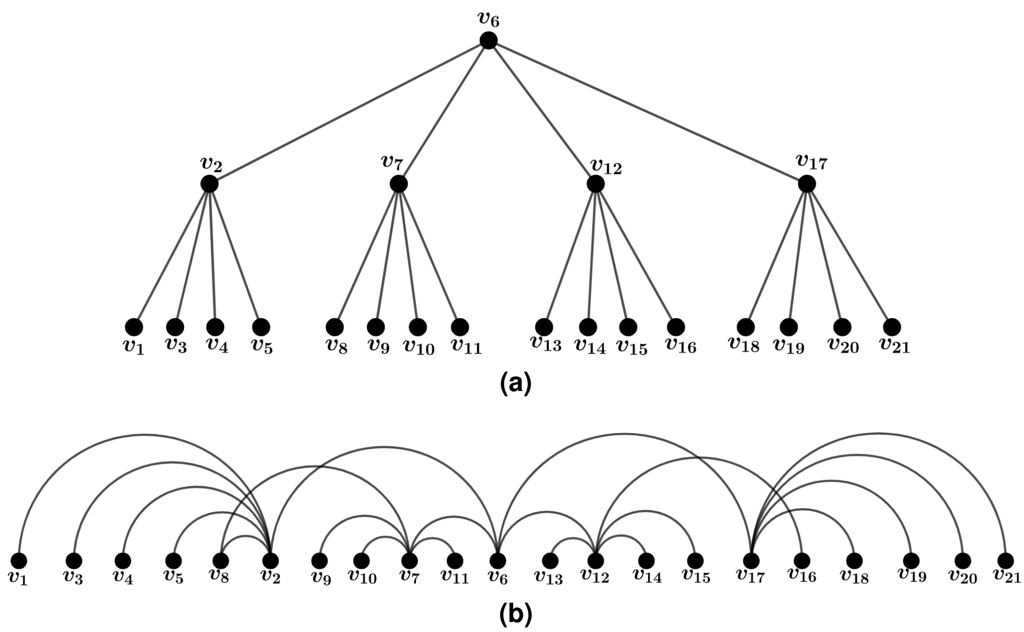

Figure 2. A graph with the optimal linear layout and $bw(G) = 5$ [Chung 1988].

3. The relationship between the Firefighter problem and the bandwidth

In this section, we show a lower bound for the $sn(G, v)$, as a function of the number of vertices and the bandwidth of G.

Theorem 1. *Let G be a graph and $v \in V(G)$. Then $sn(G, v) \geq |V(G)| - 3bw(G)^2$.*

Sketch of the proof. Let L be a linear layout with bandwidth $bw(G)$. Our strategy is based on the observation that, after defending $bw(G)$ consecutive vertices of L, if the fire is restricted to one side of the defended vertices, it cannot reach the other side.

In Figure 3, we have a simplified illustration of the strategy, in a piece of the linear layout of a graph with $bw(G) = 3$. Note that since there are no edges between vertices of distance greater than 3, the fire cannot reach vertices to the right of the vertex labeled b_2.

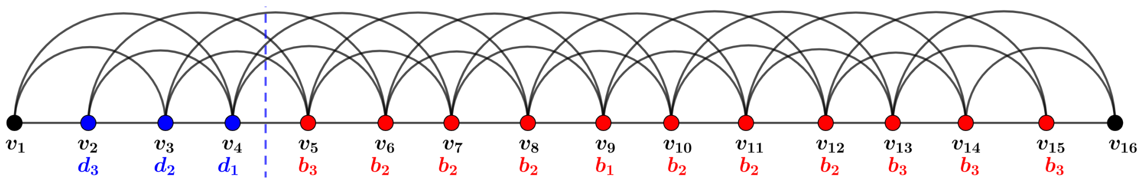

Figure 3. We can defend $bw(G)$ vertices imediately after the fire reach the vertex next to the first defended vertex, where it would be the last chance for the fire to continue propagating to the left, as illustrated in the figure. After the last defense $d_{bw(G)}$, the fire remained on both sides, and in the worst case it burned $bw(G)^2$ vertices. The same strategy will be applied while the fire spreads to the right, and at the end of it, we will have at most $3bw(G)$ burned vertices.

Let i be the position in L of the vertex v where the fire starts, i.e., $\pi(v) = i$. In the first step, v_i is the unique burned vertex. From the definition of bandwidth, after k steps the fire cannot reach a vertex of index smaller than $i - (k-1)bw(G)$ or bigger than $i + (k-1)bw(G)$ (see Figure 4).

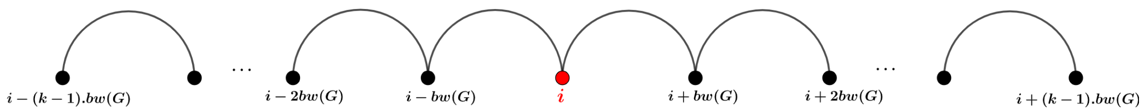

Figure 4. At the k-th step, all burned vertices have labels with a maximum diference $(k-1)bw(G)$ from the first burned vertex i. It is easy to see that after bw steps, the fire cannot reach a vertex of index smaller than $i - (bw(G)-1)bw(G)$ or bigger than $i + (bw(G)-1)bw(G)$.

The fire only starts to spread after step $k \geq 2$, that is, we are always able to defend a vertex of distance $bw(G)$ in linear layout from any burned vertex. Therefore, defending $bw(G)$ vertices is enough to form a barrier that contains the fire, as every vertex spreads the fire to a maximum of $bw(G)$ positions in the layout. This will be a crucial part of our strategy, as we will be able to defend each vertex of our barrier before the fire reaches it through an adjacent burned vertex.

We refer to Figure 3 to follow our strategy. In the first step, the fire starts at vertex i (vertex v_9 in Figure 3). We defend the vertex at position $j = i - bw(G)^2 + 2bw(G) - 2$ (vertex v_4 in Figure 3), and in the k-th step, for $1 \leq k \leq bw(G)$ we defend the vertex at position $j-k+1$. Note that, up to step $bw(G)-1$, the smallest index the fire could reach is $i-(bw(G)-2)bw(G) = i - bw(G)^2 + 2bw(G)$ (vertex v_6 in Figure 3). Hence, in the $bw(G)-1$ first steps, the defended vertices, which are in the interval $[j-(bw(G)-1)+1, j] = [i-bw(G)^2+bw(G), i-bw(G)^2+2bw(G)-2]$ (interval $[v_3, v_4]$ in Figure 3) were not reached by the fire yet. In step $bw(G)$, the fire would reach vertices in the range $[i-bw(G)^2+bw(G), i-bw(G)^2+2bw(G)-1]$ (interval $[v_3, v_5]$ in Figure 3), and only the vertex at position $i-bw(G)^2+2bw(G)-1$ will be set on fire, since the rest of the interval is defended. Finally, we defend the vertex at position $i-bw(G)^2+bw(G)-1$ and the fire cannot spread to smaller indices, since $bw(G)$ consecutive vertices are protected.

Following an analogue strategy, in steps $bw(G)+1$ to $2bw(G)$ it is possible to defend $bw(G)$ consecutive vertices to the right of i, in the interval $[i+2bw(G)^2-2bw(G)+2, i+2bw(G)^2-bw(G)+1]$, ensuring that no vertices of larger indices are reached. Hence, in the worst case all vertices in the range $[i-bw(G)^2+2bw(G)-1, i+2bw(G)^2-2bw(G)+1]$ are burned, which are less than $3bw(G)^2$ vertices, and the result follows. $\square$

4. Discussion and future work

In this paper we established a relationship between the bandwidth and the number of vertices we are able to save in the Firefighter problem. One way to think about this result is to consider graph classes of bounded bandwidth. We say that a graph class $\mathcal{C}$ has bounded bandwidth if there is a constant c such that for all $G \in \mathcal{C}$, $bw(G) \leq c$. Our result implies that for a class of bounded bandwidth $\mathcal{C}$, $sn(G, v) = \Omega(|V(G)|)$ for any $G \in \mathcal{C}$ and any $v \in V(G)$. As example, if $\mathcal{C}$ is the class P_n^k, we know that $bw(P_n^k) \leq k$ (with equality for large enough n) and it is possible to show that the bound given by Theorem 1 is tight, generalizing some previous results. However, this is not always the case, as it is not hard to build examples where we can save more vertices. An interesting research direction would be to understand under what conditions the bound can be strengthened.

References

Chung, F. (1988). Selected topics in graph theory. Academic Press.

Chvátal, V. (1970). A remark on a problem of harary. Czechoslovak Mathematical Journal, 20:109–111.

Garey, M. and Johnson, D. (1979). Computers and Intractability: A Guide to the Theory of NP-Completeness. New York: W.H. Freeman.

Hartnell, B. (1995). Firefighter! An application of domination. In The 25th Manitoba Conference on Combinatorial Mathematics and Computing.