Valid Paths, a short synthesis

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Abstract. *A labeling of a graph is an assignment of labels to its vertices and/or edges. The Path Validity Problem seeks to optimise the quantity of 2-paths with the integer label associated with the middle vertex being smaller than those associated with the path endpoints. Recent research has investigated the behaviour of optimal labeling in several classes of graphs. This work provides a summary overview of recent advances in the study of this challenging problem and presents that the decision problem of its maximising variant is $\mathcal{NP}$-Complete, while its minimising variant remains an open problem.*

1. Introduction

Introduced by [Rosa 1967], the labeling of a graph is an assignment of labels to its sets of vertices or edges (or both), usually subject to some constraints. Thousands of studies addressing this concept have been published by the year 2022 and many of them are listed in the Gallian's dynamic survey [Gallian 2023].

In this article a special type of labeling called *numbering* is the main focus of study.

[Foulds and Longo 2023] introduced the concept of *Path Validity*, which is based on the *vertex numbering* (see the first paragraph of Section 2 for a formal definition of this concept) of an arbitrary given simple connected graph G. A path in G is termed *valid* if it is a 2-path and the integer associated with the middle vertex is smaller than those at the path endpoints. The *Path Validity Problem* involves finding a numbering of a given graph G that optimises the number of valid paths in G induced by such a numbering. This work provides a summary overview of recent advances in the *Path Validity Problem* with emphasis in finding a numbering with maximum or minimum quantity of valid paths and their complexity.

This problem arises, for example, in the enumeration of chordless cycles in graphs, which has application in the prediction of nuclear magnetic resonance, chemical shift values and ecological networks [Dias et al. 2013]. Another possible application can be found in systems with intermediate validation. For instance, suppose a communication network where each node has a different rank and communication between two devices at nodes A and B must be validated/mediated by a device at a node C, connected to both A and B, with higher rank than those of A and B. Thus, the quantity of possible connections, subject to the intermediate validation, is equivalent to the number of *valid paths*. Furthermore, ranking the entire network is equivalent to numbering all of its nodes and maximising the quantity of valid connections is equivalent to maximising the quantity of *valid paths* in the support graph of the network.

2. The path validity problem

Let $G = (V(G), E(G))$ be a simple connected graph, with vertex set $V(G)$, edge set $E(G)$ and $n = |V(G)|$. A numbering π of G is a bijective mapping from $V(G)$ to $\{1, 2, \ldots, n\}$, where $\pi(v)$ denotes the number associated with vertex $v \in V(G)$. A 2-path $\langle x, u, y\rangle$ in a numbered graph G is said to be *valid* if the number associated with its middle vertex u is smaller than the numbers of its endpoints x and y, i.e., $\pi(u) < \min\{\pi(x), \pi(y)\}$.

The validity $\phi_\pi(G)$, of a numbering π of G, is defined as the quantity of valid 2-paths of G induced by π, i.e.,

$$\phi_\pi(G) = |\{\langle x, u, y\rangle \mid u \in V(G),\ x, y \in Adj(u) \text{ and } \pi(u) < \min\{\pi(x), \pi(y)\}\}|. \quad (1)$$

The *minimum validity* $\phi_{\min}(G)$ of G is the minimum of $\phi_\pi(G)$ over all possible numberings of G, i.e.,

$$\phi_{\min}(G) = \min\{\phi_\pi(G) \mid \pi \text{ is a numbering of } G\}. \quad (2)$$

Analogously, the *maximum validity* $\phi_{\max}(G)$ of G is the maximum of $\phi_\pi(G)$, i.e.,

$$\phi_{\max}(G) = \max\{\phi_\pi(G) \mid \pi \text{ is a numbering of } G\}. \quad (3)$$

For general graphs, $\phi_{\min}(G)$ and $\phi_{\max}(G)$ can vary significantly, depending on the graph structure and how the numbering is carried out. If no constraints are imposed then $n!$ different numberings are possible, inducing sets of valid paths that may vary quite markedly. Figure 1 illustrates this phenomenon for three distinct numberings of the C_6 graph, with the valid paths indicated in red. The leftmost numbering induces only a single valid path, the minimum quantity, namely $\langle 2, 1, 6\rangle$, and is a manifestation of $\phi_{\min}(C_6)$. The second numbering induces two valid paths, namely $\langle 4, 1, 5\rangle$ and $\langle 3, 2, 5\rangle$. The rightmost numbering induces the maximum quantity, a triple, of valid paths, namely $\langle 4, 1, 6\rangle$, $\langle 4, 2, 5\rangle$ and $\langle 5, 3, 6\rangle$, and is a manifestation of $\phi_{\max}(C_6)$.

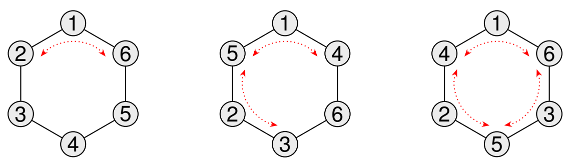

Figure 1. Three numberings of the C_6 graph.

The problems of finding numberings that achieve $\phi_{\min}(G)$ or $\phi_{\max}(G)$ for an arbitrary graph G, denoted by MIN-VP(G, k) and MAX-VP(G, k), respectively, are defined as:

Instance:	A finite, undirected, simple graph G and a nonnegative integer k.
MIN-VP(G, k):	Is there a numbering π of G such that $\phi_\pi(G) \leqslant k$?
MAX-VP(G, k):	Is there a numbering π of G such that $\phi_\pi(G) \geqslant k$?

As far as we know, the $\mathcal{NP}$-completeness of MIN-VP(G, k), stated by Conjecture 2.1, is an open problem. On the other hand, the $\mathcal{NP}$-completeness of MAX-VP(G, k) is stated by Theorem 2.2. A proof of this theorem was recently obtained [Leal 2024] with a reduction from a node-deletion problem stated by [Krishnamoorthy and Deo 1979], which is valid even for regular graphs of degree 3 (Cubic Graphs).

Conjecture 2.1 *MIN-VP(G, k) is an $\mathcal{NP}$-complete problem.*

Theorem 2.2 *MAX-VP(G, k) is an $\mathcal{NP}$-complete problem.*

3. Related Works

[Foulds and Longo 2023] and [Leal et al. 2023] investigated the behaviour of $\phi_{\min}(G)$ and $\phi_{\max}(G)$ on some graph classes and provided the results summarised in Table 1. Those authors observed that apparently there is no recognisable general pattern for a numbering that leads to the optimisation (minimisation or maximisation) of the quantity of valid paths (each graph class in the table has its own individual pattern). [Leal et al. 2023] also proposed two non-polynomial algorithm to find a numbering which induces $\phi_{\min}(G)$ and $\phi_{\max}(G)$.

Table 1. $\phi_{\min}(G)$ and $\phi_{\max}(G)$ for some classes of graphs.

G	$\phi_{\min}(G)$	$\phi_{\max}(G)$
P_n	0	$\lceil \frac{n}{2} \rceil - 1$
T_n	0	$\binom{n-1}{2}$
C_n	1	$\left\lceil \frac{n-1}{2} \right\rceil$
W_n	n	$\binom{n-1}{2} + \left\lceil \frac{n-1}{2} \right\rceil$
$K_{p,q}$	$(3q-p-1)(p^2-p)/6$	$p\binom{q}{2}$
K_n	$\sum_{i=2}^{n-1} \binom{i}{2}$	$\sum_{i=2}^{n-1} \binom{i}{2}$
$G_{n,n}$	$(n-1)^2$	$3(n-1)^2$
A_n $(n=5)$	$3n-8$	$3n-5$
A_n $(n=6)$	$3n-8$	$3n-1$
A_n $(n \geqslant 7)$	$3n-8$	$[5n-12, 5n-11]$
$PC_{n,r}$	$\binom{2\|r\|+1}{3} + (n-2\|r\|-1)\binom{\|r\|}{2}$	–
$SC_{n,r}$	–	$\binom{2\|r\|}{3} + \left(\frac{n}{2} - \|r\|\right)\binom{2\|r\|-1}{2}.$

The notation used in Table 1 concerns the following graphs of finite order n: P_n is a linear graph ($n \geqslant 3$); T_n is a connected, acyclic n-graph; C_n is a chordless, closed n-path (cycle graph); W_n is a wheel graph (the Cartesian product of a solitary vertex and a cycle graph); $K_{p,q}$ is a complete bipartite graph, with vertex set partition of finite orders p and q ($1 \leqslant p \leqslant q$); K_n is the complete graph; $G_{n,n}$ is a square grid, n odd (n is a perfect square and the number of rows and columns in the grid are both $\sqrt{n}$); A_n is an Apollonian graph (graphs constructed from a single triangle embedded in the Euclidean plane, by repeatedly adding a new vertex within a triangular face and connecting it to the vertices of the face).

Circulant graphs $C_n(R)$ are graphs whose binary circulant adjacency matrix is specified by a connection set R. $PC_n(R)$ and $SC_n(R)$ are prefix and suffix circulant graphs, where R is, respectively, a prefix (suffix) contiguous subset of the set $\{1, 2, \ldots, \lceil \frac{n}{2} \rceil - 1\}$, i.e., $R = \{1, 2, \ldots, k_p,\ 1 \leqslant k_p \leqslant \lceil \frac{n}{2} \rceil - 1\,\}(R = \{k_s, k_s + 1, \ldots, \lceil \frac{n}{2} \rceil - 1,\ 1 \leqslant k_s \leqslant \frac{n}{2} \rceil - 1)$.

4. Conclusion

The *Valid Path problem* involves finding a numbering that optimises the quantity of valid paths in a given graph (see Section 2). This work compiles the studies related to valid paths done in the past few years and presents a new application related to the problem. These studies showed that it is a hard problem, with no general labeling pattern, and presented some outcomes on particular classes of graphs. The MAX-VP(G, k) version of the problem belongs to the $\mathcal{NP}$-complete class (Theorem 2.2), a recent finding, that promotes the importance of the problem.

The main focus of future works is to complete the Table 1 with both $\phi_{\min}(G)$ and $\phi_{\max}(G)$ behaviours, since some classes are incomplete. Another objective is proving the Conjecture 2.1.

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