On the AVD-total chromatic number of circulant graphs

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Abstract. *AVD-k-total coloring of a simple graph G is a mapping $\pi : V(G) \cup E(G) \to \{1, \dots, k\}$ such that: adjacent or incident elements $x, y \in V(G) \cup E(G)$, $\pi(x) \neq \pi(y)$; and for each pair of adjacent vertices $x, y \in V(G)$, sets $\{\pi(x)\} \cup \{\pi(xv) \mid xv \in E(G)$ and $v \in V(G)\}$ and $\{\pi(y)\} \cup \{\pi(yv) \mid yv \in E(G)$ and $v \in V(G)\}$ are distinct. The AVD-total chromatic number, denoted by $\chi''_a(G)$ is the smallest k for which G admits an AVD-k-total-coloring. [Zhang et al. 2005] conjectured that any graph G has $\chi''_a(G) \leq \Delta+3$. [Hulgan 2009] conjectured that any subcubic graph G has $\chi''_a(G) \leq 5$. In this article, we proved that all cubic circulant graph has $\chi''_a(C_{2n}(d, n))) = 5$, being positive evidence to Hulgan's conjecture.*

1. Introduction

Let $G = (V, E)$ be a simple connected graph and Δ be the maximum degree of G. A *k-total coloring* of G is an assignment of k colors to the vertices and edges of G so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta + 1$. The Total Coloring Conjecture (TCC) states that $\chi''(G) \leq \Delta + 2$, for every simple graph [Behzad 1965, Vizing 1964]. Graphs with $\chi''(G) = \Delta + 1$ are called *Type 1*, and graphs with $\chi''(G) = \Delta + 2$ are called *Type 2*. It is known that determining the total chromatic number is an NP-complete problem [McDiarmid and Sánchez-Arroyo 1994].

Let π be a k-total coloring of G and let $C_\pi(u) := \{\pi(u)\} \cup \{\pi(uv) \mid uv \in E(G), v \in V(G)\}$ be the set of colors that *occurs* in a vertex $u \in V(G)$. If it is clear from the context that π is a k-total coloring of G, then $C_\pi(u)$ is written simply as $C(u)$. We denote by $\overline{C}(u)$ the set of colors of $\{1, \dots, k\}$ that do not occur in vertex $u \in V(G)$. Two vertices u and v are *distinguishable* when $C(u) \neq C(v)$. If this property is true for every pair of adjacent vertices, then π is an *Adjacent-Vertex-Distinguishing-k-Total-Coloring*, or simply *AVD-k-total coloring*. The *AVD-total chromatic number* of G, denoted by $\chi''_a(G)$, is the smallest k for which G admits an AVD-k-total coloring. [Zhang et al. 2005] introduced the AVD-total coloring problem. The authors determined the AVD-total chromatic number for some families of simple graphs and noted that all of them admit an AVD-total coloring with at most $\Delta(G) + 3$ colors. Based on these results, the authors posed the Conjecture 1.1, the AVD-Total Coloring Conjecture (AVD-TCC).
Conjecture 1.1 ([Zhang et al. 2005]). *If G is a simple graph, then $\chi''_a(G) \leq \Delta(G) + 3$.*

If AVD-TCC holds, we can classify any graph between three sets according to the AVD-total chromatic number. If $\chi''_a(G) = \Delta(G) + 1$, then G is called *AVD-Type 1*. If

$\chi''_a(G) = \Delta(G) + 2$, then G is called *AVD-Type 2*. If $\chi''_a(G) = \Delta(G) + 3$, then G is called *AVD-Type 3*. It is straightforward that any *AVD-Type 1* graph is *Type 1* graph. However, the reverse is not true. For instance, the complete graph K_n with n odd, is *Type 1* and *AVD-Type 3* [Yap 1996, Chen and Zhang 2008].

Since the proposal of this conjecture, several studies have been conducted. [Chen 2008] proved the AVD-TCC to subcubic graphs. [Chen and Guo 2009], determined the AVD-total chromatic number of hypergraphs Q_n. In the same year, [Hulgan 2009] presented concise proofs of AVD-Total chromatic number of cycles, complete graphs, and subcubic graphs. Furthermore, it was questioned whether the upper bound for the AVD-total chromatic number of subcubic graphs is sharp. Thus, Conjecture 1.2 was proposed.

Conjecture 1.2 ([Hulgan 2009, Hulgan 2010]). *If G is a simple graph with $\Delta(G) = 3$, then $\chi''_a(G) \leq 5$.*

Since then, recent studies have been conducted involving some other graph classes to investigate the Conjecture 1.1, such as equipartite graphs, split graphs, corona graphs, and 4-regular graphs [Luiz et al. 2015, Papaioannou and Raftopoulou 2014, Verma and Panda 2022, Verma et al. 2022]. To investigate the Conjecture 1.2, we highlight the work conducted by [Luiz et al. 2017], that proved that the Conjecture 1.2 holds for some infinite families of subcubic graphs, where vertices with a maximum degree are not adjacent, and for infinite families of Snarks [1] such as flower snarks, generalized Blanuša snarks, LP_1-snarks.

A *circulant graph* $C_n(d_1, d_2, \cdots, d_\ell)$ with integers numbers $1 \leq d_i \leq \lfloor n/2 \rfloor$, where $1 \leq i \leq \ell$ and $\ell \leq \lfloor n/2 \rfloor$, has vertex set $V = \{v_0, v_1, \cdots, v_{n-1}\}$ and edge set $E = \bigcup_{i=1}^{\ell} E_i$, where $E_i = \{e_0^i, e_1^i, \cdots, e_{n-1}^i\}$ and $e_j^i = v_j v_{j+d_i}$ (if n is even and $d_\ell = n/2$, then $E_\ell = \{e_0^\ell, e_1^\ell, \ldots, e_{\frac{n-2}{2}}^\ell\}$), where the indexes of the vertices are considered modulo n. An edge of E_i has length d_i. Some classical circulant graphs, such as the cycle graphs $C_n \simeq C_n(1)$, the complete graphs $K_n \simeq C_n(1, 2, ..., \lfloor n/2 \rfloor)$ have their AVD-total chromatic number determined by [Zhang et al. 2005]. Some partial results are known to power of cycle graphs, the infinite family of circulant graphs $C_n^k := C_n(1, 2, ..., k)$. [Alvarado et al. 2019] proved that C_n^2 and C_n^k with $n \equiv 0 \mod (k+1)$ are *AVD-Type 2*.

It is well known that the connected cubic circulant graph is written as $C_{2n}(1, n)$, which is known as *Möbius ladder* or $C_{2n}(2, n)$ and n is odd in this case, which is known as *circular ladder* [Hackmann and Kemnitz 2004]. It was proved that the circular ladder graphs are *AVD-Type 2* [Zhu et al. 2016].

In this paper, we proved that all cubic circulant graphs $C_{2n}(d, n)$ are *AVD-Type 2*, i.e., $\chi''_a(C_{2n}(d, n)) = 5$, which is positive evidence to Conjecture 1.2.

2. Main result

In this section, we prove that $C_{2n}(d, n)$ is *AVD-Type 2* (see Theorem 2.1). To establish it, we use two auxiliary properties to apply to the case where the circulant graph is bipartite.

Proposition 2.1 ([Zhang et al. 2005]). *Let G be a simple graph. If G has two adjacent vertices of maximum degree, then $\chi_a''(G) \geq \Delta(G) + 2$. On the other hand, if G does not have adjacent vertices of maximum degree, then $\chi_a''(G) \geq \Delta(G) + 1$.*

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[1] Snarks are bridgeless cubic graphs with chromatic index equal to 4.

Proposition 2.2 ([Chen and Zhang 2008]). *If G is a bipartite graph, then $\chi_a''(G) \leq \Delta(G) + 2$.*

The following lemmas characterize bipartite circulant graphs (Lemma 2.1) and cubic circulant graphs (Lemma 2.2).

Lemma 2.1 ([Heuberger 2003]). *The circulant graph $C_n(d_1, d_2, \ldots, d_\ell)$ is bipartite if and only if n is even and d_i is odd, for all $i \in \{1, 2, \ldots, \ell\}$.*

Lemma 2.2 ([Hackmann and Kemnitz 2004]). *If $l = \gcd(d, n)$ such that $d = lm$ and $n = lp$, then $C_{2n}(d, n)$ is isomorphic either to l copies of $C_{2p}(1, p)$ if m is odd or to l copies of $C_{2p}(2, p)$ if m is even.*

Theorem 2.1. *Every cubic circulant graph $C_{2n}(d, n)$ is AVD-Type 2.*

Proof. Let G be a cubic circulant graph, i.e, $G \simeq C_{2n}(d, n)$. From Lemma 2.2, G is isomorphic to $l = \gcd(d, n)$ copies of $C_{2n}(1, n)$ or l copies of $G \simeq C_{2n}(2, n)$. So, it is sufficient to prove that a connected component of G is *AVD-Type 2* to guarantee that G is *AVD-Type 2*. Then G is either $C_{2n}(1, n)$ or $C_{2n}(2, n)$.

1. Suppose that $G \simeq C_{2n}(1, n)$. From Proposition 2.1, $\chi_a''(G) \geq 5$.
   (a) If n is odd, from Lemma 2.1, G is bipartite and $\chi''_a(G) \leq 5$. Therefore, $\chi''_a(G) = 5$.
   (b) If n is even, then we construct an AVD-5-total coloring π of G as on Figure 1:
   - to the vertices v_i and edges v_iv_{i+1} with $i \in \{0, 1, \ldots, n-1\}$:

$$\pi(v_i) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd.} \end{cases} \qquad \pi(v_iv_{i+1}) = \begin{cases} 3, & \text{if } i \text{ is even;} \\ 4, & \text{if } i \text{ is odd.} \end{cases} \tag{1}$$

   - to the vertices v_i and edges v_iv_{i+1} with $i \in \{n, n+1, \ldots, 2n-1\}$:

$$\pi(v_i) = \begin{cases} 3, & \text{if } i \text{ is even;} \\ 4, & \text{if } i \text{ is odd.} \end{cases} \qquad \pi(v_iv_{i+1}) = \begin{cases} 1, & \text{if } i \text{ is even;} \\ 2, & \text{if } i \text{ is odd.} \end{cases} \tag{2}$$

   - to the edges v_iv_{i+n} for all $i \in \{0, 1, \ldots, n\}$:

$$\pi(v_iv_{i+n}) = 5 \tag{3}$$

   Therefore, we will display that π is a 5-total coloring of G. Notice that (3) is a perfect matching M of G. So we use the colors 1, 2, 3, and 4 to the vertices and edges of $G[V \setminus M]$. Moreover, $G[V \setminus M]$ is an even cycle with size $2n$. We split it cycle on paths $P_1 := (v_0v_1 \ldots v_{n-1})$ and $P_2 := (v_nv_{n+1} \ldots v_{2n-1})$ and edges $v_{n-1}v_n$ and $v_{2n-1}v_0$.
   From (1), we alternate to the path P_1 the colors 1 and 2 to the vertices and 3 and 4 to the edges; and from (2), we alternate to the path P_2 the colors 3 and 4 to the vertices and 1 and 2 to the edges. Since the adjacent vertices of P_1 are alternate with colors 1 and 2, adjacent vertices v_i and v_{i+1} have different colors. Futhermore, since the edges are alternate with colors 3 and 4, adjacent edges v_iv_{i+1} and $v_{i+1}v_{i+2}$ have different colors. Also, v_{i+1} have different colors of v_i and v_{i+2}. The same argument can be made to the path P_2. Since $\pi(v_i) = 1$ and $\pi(v_{i+n}) = 3$ if i is even with

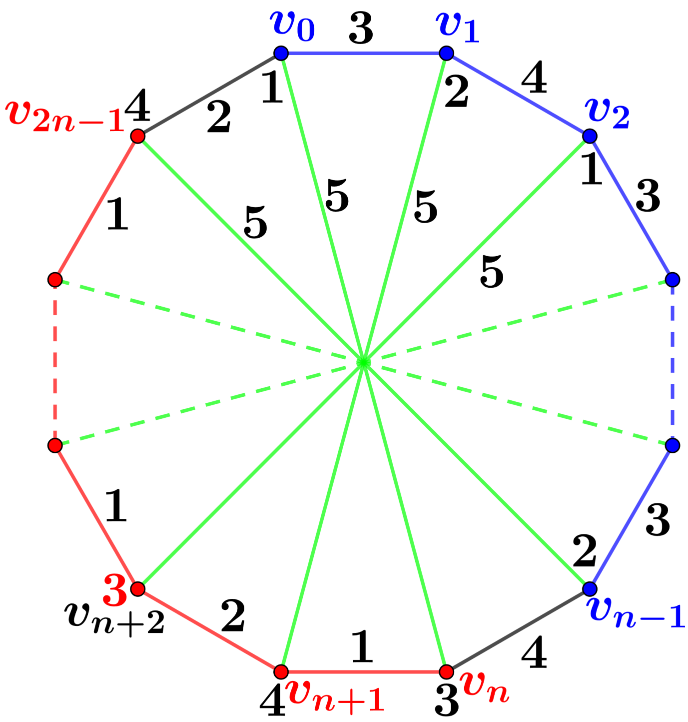

Figure 1. A scheme of the AVD-5-total coloring π of $C_{2n}(1, n)$. The elements in blue denote the elements in the path P_1 and the elements in red denote the elements in the path P_2. The set of edges in green is a perfect matching.

$i \in \{0, 1, \ldots, n-1\}$; and $\pi(v_i) = 2$ and $\pi(v_{i+n}) = 4$ if i is odd with $i \in \{0, 1, \ldots, n-1\}$. Therefore, $\pi(v_i) \neq \pi(v_{i+n})$. Finally, we proved that π is an AVD-5-total coloring of G. Notice that if a vertex v_i in P_1 has $1 \in C(v_i)$, then $1 \notin C(v_{i+1})$ for $i \in \{0, 1, \ldots, n-2\}$. For $i = n-1$, $2 \in C(v_i)$ and $2 \notin C(v_{i+1})$, then $C(v_i) \neq C(v_{i+1})$ for $i \in \{0, 1, \ldots, n-1\}$. The same occurs to the path P_2, if $3 \in C(v_i)$, then $3 \notin C(v_{i+1})$, for $i \in \{n, n+1, \ldots, 2n-2\}$. For $i = 2n-1$, $4 \in C(v_i)$ and $4 \notin C(v_{i+1})$ and so $C(v_i) \neq C(v_{i+1})$, for $i \in \{n, n+1, \ldots, 2n-2\}$. Finally for $i = 0$, $2 \in C(v_i)$ and $2 \notin C(v_{i+n})$; for $i \in \{1, 2, \ldots, n-2\}$, $3 \in C(v_i)$ and $3 \notin C(v_{i+n})$ if i is odd, and $4 \in C(v_i)$ and $4 \notin C(v_{i+n})$ if i is even. Hence, any pair of adjacent vertices $u, v \in V(G)$, has $C(u) \neq C(v)$, and π is an AVD-5-total coloring of G.

2. Suppose that $G \simeq C_{2n}(2, n)$. We remark that G is the circular ladder, a generalized Petersen graph. From [Zhu et al. 2016], $C_{2n}(2, n)$ is *AVD-Type 2*.

Hence, G is *AVD-Type 2* graph. □

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