Generalizing the coloring game from caterpillars to trees*

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Abstract. *The coloring game is a two-player non-cooperative game conceived in 1981. Alice and Bob alternate turns to properly color the vertices of a finite graph G with t colors. Alice's goal is to properly color the vertices of G with t colors; Bob's aim is to prevent it. If, at any point, there is an uncolored vertex without an available color, Bob wins; otherwise, Alice wins. The game chromatic number $\chi_g(G)$ is the smallest t for Alice to have a winning strategy. In 1991, Bodlaender showed that a caterpillar was the smallest tree T with $\chi_g(T) = 4$; in 1993, Faigle et al. proved $\chi_g(T) \leq 4$ for every tree T. In 2015, Dunn et al. proposed the characterization of forests with game chromatic numbers 3 and 4. In this paper, we extend results from caterpillars to more general trees, and establish sufficient conditions to ensure that a tree has game chromatic number 4.*

1. Introduction

Let $G = (V, E)$ be a finite, simple and undirected graph, with vertex set $V = V(G)$ and edge set $E = E(G)$. In 1981, Martin Gardner [7] published for the first time a two player non-cooperative map-coloring game created by Steven Brams. A decade later, this game was reinvented by Bodlaender [1], who adapted it to the context of graphs, and named it the coloring construction game, which was later renamed the *coloring game.* The game involves two players, Alice and Bob, who alternate turns to properly color the uncolored vertices of a graph G using colors in a given color set with t colors. Alice's goal is to color graph G with t colors, and Bob does his best to prevent it. Alice wins when all vertices are properly colored with t colors; otherwise, Bob wins.

We define Alice as having a winning strategy with t colors when she has a sequence of moves that ensures that the graph can be completely (properly) colored with t colors regardless of Bob's moves along the game. Analogously, we say that Bob has a winning strategy with t colors when he has a sequence of moves that ensures that the graph can not be completely (properly) colored with t colors regardless of Alice's moves during the game.

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The *game chromatic number* $\chi_g^a(G)$ (or simply $\chi_g(G)$) of G is the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when she starts the game. Building upon the previous research by Dunn et al. [3] and Furtado et al. [5, 6], we investigate $\chi_g^b(G)$, the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Bob starts the game. In case of equality, we use the notation $\chi_g^{a,b}(G) = \chi_g^a(G) = \chi_g^b(G)$. This means that Alice has a winning strategy for the graph coloring game on G with $\chi_g^{a,b}(G)$ colors, regardless of who started playing.

Let G be a graph, Z a set of vertices of G, $c : Z \to \{1, \ldots, t\}$ a coloring function that assigns to each vertex $v \in Z$ a color $c(v)$, and (G, Z, c) be the partially colored graph. We say that Alice (resp. Bob) plays on (G, Z, c), if Alice (resp. Bob) colors the uncolored vertices in $V(G) \setminus Z$. We introduce the auxiliary parameter $\chi_g^a(G, Z, c)$ (resp. $\chi_g^b(G, Z, c)$) as the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Alice (resp. Bob) starts playing on a graph G with a previously colored set of vertices $Z \subseteq V(G)$ by coloring function c. In order to simplify the notation, we omit the coloring c and write $\chi_g^a(G, Z)$ (resp. $\chi_g^b(G, Z)$) to represent $\chi_g^a(G, Z, c)$ (resp. $\chi_g^b(G, Z, c)$).

The coloring game has been extensively studied for various graph classes to obtain improved upper and lower bounds for $\chi_g(G)$, including toroidal grids [11], Cartesian products of certain graph classes [2], planar graphs [12], outerplanar graphs [10], forests [3], and partial k-trees [13].

Our research contribution focuses on analyzing the game in arbitrary trees that contain a specific type of graph known as caterpillars. A *caterpillar* $H = cat(k_1, k_2, \ldots, k_s)$ is a tree obtained from a central path v_1, v_2, v_3, …, v_s (called *spine*) by joining k_i leaf vertices to v_i, for each $i \in \{1, \ldots, s\}$; and with number of vertices $n = s + \sum_{i=1}^{s} k_i$.

Caterpillars were initially investigated by Bodlaender [1], who defined an infinite family of caterpillar trees with a game chromatic number of 4. Moreover, Dunn et al. [3] established a criteria for determining the game chromatic number of a forest without vertices of degree 3. In particular, we focus on studying a special category of caterpillars denoted as 1-*caterpillars* defined by $H = cat(k_1, ..., k_s)$ that are caterpillars with $k_1 = k_s = 0$, and for $i \in \{2, \ldots, s-1\}$, $k_i = 1$, with spine vertices v_1, v_2, v_3, …, v_s, and respective leg leaves $\lambda_2, \lambda_3, \ldots, \lambda_{s-1}$ (where λ_i is the vertex adjacent to v_i). By definition, this caterpillar have maximum degree 3, making them incompatible with the criteria established by Dunn et al. [3].

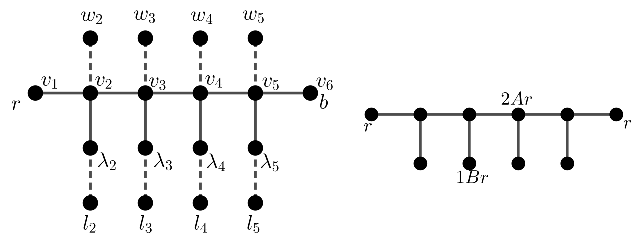
The motivation of our present work arises from two references concerning the coloring game on 1-caterpillars. In an abstract [8], related to his thesis in French [9], Guignard claimed to have a characterization of the game chromatic number for 1-caterpillars. However, despite his significant contribution to the study of the game chromatic number of trees, it appears that Guignard's work has not received the attention it deserves due to the lack of subjecting his work to a rigorous peer review process. On the other hand, independently, Furtado et al. in [5] established conditions to guarantee that 1-caterpillars have game chromatic number 4. We observe that Furtado et al. [5] infinite family of 1-caterpillars aligns with the game chromatic number claimed by Guignard [8]. We hope that our research will motivate further exploration of Guignard's findings in his doctoral thesis and encourage comparison with our current work, which is an extension of part of his work for trees containing caterpillars.

2. Caterpillars in trees

The works of Guignard in [8] and Furtado et al. [5] while employing different techniques converge on the study of several cases for a caterpillar with a previously colored set of vertices that leads Bob's victory in the game with 3 colors. We modify the techniques previously utilized by Furtado et al. [5] to suit the context of trees. Next, we provide an overview of the key definitions and results necessary for this analysis.

Let T be a tree, H' be a caterpillar, and H'' an induced subgraph of T isomorphic to H'. Consider two previously colored sets Z and Z' with $Z \subseteq V(T)$ and $Z' \subseteq V(H')$. We say that a game on (T, Z) has a *copy of the game* on (H', Z') (or simply a *copy* of (H', Z')), if there exists an isomorphism $\phi : V(H') \to V(H'')$ that preserves the colors of the vertices of H' (both colored and uncolored) in the vertices of H''.

We consider games (T, Z) that have a copy of a game (H', Z') where H' is a 1-caterpillar and Z' is a previously colored set of vertices. Let (H'', Z'') be the copy of the game (H', Z') in (T, Z) where T is a tree and H' a 1-caterpillar with s spine vertices, $k_i \leq 1$ for $1 < i < s$ and $Z'' \subseteq Z$. We call L the set of vertices adjacent to the leg leaves of H''. In particular, we call $l_i \in V(T)$ any vertex adjacent to λ_i, for $2 \leq i \leq s-1$. Analogously, we call W the set of vertices adjacent to the spine vertices of H''. We denote by $w_i \in V(T)$ any vertex adjacent to v_i, for $2 \leq i \leq s-1$. In our figures, we use dashed lines to represent edges that connect vertices of the copy to vertices in L or W, see Figures 1(a) and 2. Moreover, we use the notation $iA = u$ (resp. $iB = u$) to indicate that Alice (resp. Bob) colors the vertex u in the i-th turn. When detailing a game, we denote color 1 as r (red), color 2 as b (blue), and color 3 as g (green). Thus, $iAr = u$ (or $iBr = u$) signifies that Alice (or Bob) colors vertex u with the color red on the i-th turn. Similarly for colors b and g, see Figure 1(b).



(a) (b)

Figure 1. (a) dashed lines represent edges that connect vertices of the copy to vertices in L or W; (b)$iA = w$ (resp. $iB = u$) indicates that Alice (resp. Bob) colors the vertex u in the i-th turn.

2.1. Intermittent Caterpillars:

The first kind of graphs that we study is defined as follows. First, we define the game $(\tilde{H}_s, Z')$ on a caterpillar $\tilde{H}_s$ used in Lemma 1: (i) $\tilde{H}_s = cat(k_1, \ldots, k_s)$ with s odd; (ii)

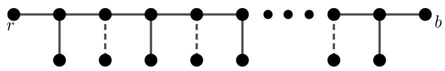


Figure 2. The partially colored intermittent caterpillar $\tilde{H}_s$, with s odd and $s \geq 5$.

$k_i = 0$, for i odd, and $k_i = 1$, for i even, $1 \leq i \leq s$. We denote this caterpillar as *intermittent caterpillar*; (iii) $Z' = \{v_1, v_s\}$, with $c(v_1)$, $c(v_s)$ can eventually be equal.

The next result describes a game that consistently appears when playing on trees that have a copy of a 1-caterpillar. Specifically, this result is crucial for proving Lemma 2.

Lemma 1. *Let T be a tree, and Z be a previously colored set of vertices of T. If (T, Z) has a copy of $(\tilde{H}_s, Z')$, for odd $s \geq 5$, and $L \cap Z = \emptyset$ or $L \cap Z \subset \{l_2, l_{s-1}\}$ with $c(l_2) = c(v_1)$ and $c(l_{s-1}) = c(v_s)$, then $\chi_g^b(T, Z) = 4$.*

2.2. 1-Caterpillars

In the following results, we study 1-caterpillars that appear as an induced subgraphs of a tree, according to s being odd or even, and having some previously colored vertices. Among our results, we establish conditions for the vertices in W and L to ensure that $\chi_g^b(T, Z) = 4$ where T is a tree and a copy of a game (H', Z') of one the 1-caterpillars with a previously colored set of vertices Z', in Figures 3(a) and 3(b) and these graphs appear in a tree T as an induced subgraph.

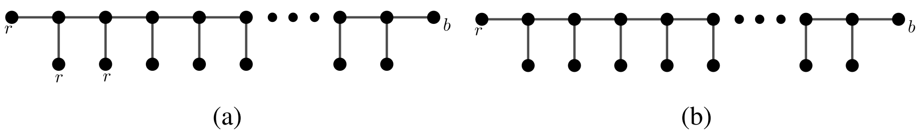


Figure 3. (a) The partially colored 1-caterpillar H', with s odd and $s \geq 9$ and $Z' = \{v_1', \lambda_2', \lambda_3', v_s' \mid c(v_1') = c(\lambda_2') = c(\lambda_3')\}$; (b)The partially colored 1-caterpillar H', with s even and $s \geq 12$.

In the following result we study a case where Bob forces the use of a fourth color and wins the game regardless of who starts playing. We heavily rely on this result in the proof of our main result, Theorem 3.

Lemma 2. *Let T be a tree, and Z be a previously colored set of vertices of T. Suppose that (T, Z) has a copy of a game (H', Z') where H' is a 1-caterpillar, with $s = 16$, and $Z' = \{v_1', v_{16}' \mid c(v_1') \neq c(v_{16}')\}$. If either $(L \cup W) \cap Z = \emptyset$, or $(L \cup W) \cap Z = \{l_2\}$ and $c(l_2) = c(v_1)$, then $\chi_g^{a,b}(T, Z) = 4$.*

3. The game on Caterpillars as induced subgraph of a tree

Theorem 3. *Let T be a tree that has a 1-caterpillar H with $s \geq 46$ as induced subgraph, then $\chi_g^{a,b}(T) = 4$. If $s \geq 31$, then $\chi_g^b(T) = 4$.*

The proof strategy for the theorem is based on the observation that a 1-caterpillar H with $s \geq 46$ contains 3 (which only overlap in a vertex two by two) copies of the 1-caterpillar with $s = 16$. Therefore, Bob's strategy in this game consists in coloring the extreme vertices of one of these copies, satisfying the conditions of the colored vertices established in Lemma 2. Analogously, a caterpillar with $s = 31$ contains 2 (which only overlap in a vertex two by two) copies of the 1-caterpillar with $s = 16$, so as Bob starts to play in these copies he can produce again a copy of the game in Lemma 2.

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