# On Tuza's conjecture for graphs with treewidth at most 6


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***Abstract.*** *Tuza (1981) conjectured that the size $\tau(G)$ of a minimum set of edges that meets every triangle of a graph $G$ is at most twice the size $\nu(G)$ of a maximum set of edge-disjoint triangles of $G$. In this paper we verify this conjecture for graphs with treewidth at most* $6$.


## 1. Introduction

In this paper, all graphs considered are simple and the notation and terminology are standard. A *triangle transversal* of a graph $G$ is a set of edges of $G$ whose deletion results in a triangle-free graph; and a *triangle packing* of $G$ is a set of edge-disjoint triangles of $G$. We denote by $\tau(G)$ (resp. $\nu(G)$) the size of a minimum triangle transversal (resp. triangle packing) of $G$. In [Tuza 1981] the following conjecture was posed:

**Conjecture** (Tuza, 1981)**.** *For every graph $G$, we have $\tau(G) \leq 2\nu(G)$.*

This conjecture was verified for many classes of graphs, in particular for planar graphs in [Tuza 1990], and tripartite graphs in [Haxell and Kohayakawa 1998]. A set of tools for dealing with graphs that contain vertices of small degree was introduced in [Puleo 2015] (Lemma 4), and Tuza's Conjecture was verified for graphs with maximum average degree less than 7, i.e., for graphs in which every subgraph has average degree less than 7. In this paper, we extend this technique (Lemma 5) in order to prove Tuza's Conjecture for graphs with treewidth at most 6 (Theorem 6). The following example (Figure 1) shows that there are graphs with treewidth at most 6 whose maximum average degree is at least 7. So our result is not implied by the previous ones. (Conversely, note that the $k \times k$ grid has treewidth at least $k$ and maximum average degree at most $4$.) Due to space limitations, we present only a sketch of some proofs.

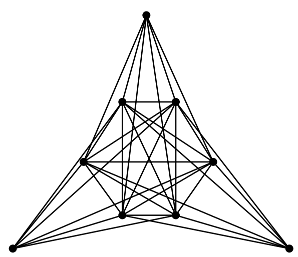

**Figure 1. A graph with treewidth 6 and average degree 22/3.**

A *rooted tree* is a pair $(T, r)$, where $T$ is a tree and $r$ is a vertex of $T$. Given $t \in V(T)$, let $P_T(t)$ be the unique path in $T$ that joins $r$ and $t$. If $t'$ is a vertex in $V(P_T(t))$, then $t'$ is an *ancestor* of $t$. Every vertex in $T$ that has $t$ as its ancestor is called a *descendant* of $t$. If $t \neq r$, then the *parent* of $t$, denoted by $p(t)$, is the ancestor of $t$ that is adjacent to $t$. The *successors* of $t$ are the vertices whose parent is $t$, and we denote the set of sucessors of $t$ by $S_T(t)$. The *height* of $t$, denoted by $h_T(t)$, is the length of a longest path that joins $t$ to a descendant of $t$. When $T$ is clear from the context, we simply write $S(t)$ and $h(t)$. For a graph $G$ and a vertex $v$ of $G$, we denote by $N_G(v)$ the set of neighbors of $v$ in $G$. When $G$ is clear from the context, we simply write $N(v)$.

A *tree decomposition* of a graph $G$ is a pair $(T, \mathcal{V})$, consisting of a tree $T$ and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of *bags* $V_t \subseteq V(G)$, satisfying the following conditions:

(T1) $\bigcup_{t \in V(T)} V_t = V(G)$;
(T2) for every $uv \in E(G)$, there exists a bag $V_t$ such that $u, v \in V_t$;
(T3) if a vertex $v$ is in two different bags $V_{t_1}, V_{t_2}$, then $v \in V_t$ for every $t$ in the path of $T$ that joins $t_1$ and $t_2$.

The *width* of $(T, \mathcal{V})$ is the number $\max\{|V_t| - 1 : t \in V(T)\}$, and the *treewidth* $tw(G)$ of $G$ is the minimum width of any tree decomposition of $G$. We refer to the vertices of $T$ as *nodes*. If $G$ is a graph with treewidth $k$, then we say that $(T, \mathcal{V})$ is a *full tree decomposition* of $G$ if $|V_t| = k + 1$ for every $t \in V(T)$, and $|V_t \cap V_{t'}| = k$ for every $tt' \in E(T)$. Every graph has a full tree decomposition (see [Bodlaender 1998, Gross 2014]).

We say that a triple $(\mathcal{V}, T, r)$ is a *rooted tree decomposition* of a graph $G$ if $(\mathcal{V}, T)$ is a full tree decomposition of $G$, $(T, r)$ is a rooted tree, and $V_t \cap V_{p(t)} \neq V_t \cap V_{t'}$ for every $t \in V(T) \setminus \{r\}$ and $t' \in S(t)$. Given a rooted tree decomposition $(\mathcal{V}, T, r)$ of a graph $G$, and a node $t \in V(T)$ with $t \neq r$, we say that the (unique) vertex $v_t \in V_t \setminus V_{p(t)}$ is the *representative* of $t$.

**Proposition 1.** *Every graph has a rooted tree decomposition.*

**Proposition 2.** *If $t$ is a leaf of a rooted tree decomposition of a graph $G$ and $y$ is the representative of $t$, then $N_G(y) \subseteq V_t$.*

## 2. Graphs with treewidth at most 6

A nonempty set $V_0 \subseteq V(G)$ is called *reducible* if there is a set $X \subseteq E(G)$ and a set $Y$ of edge-disjoint triangles in $G$ such that the following conditions hold: $(i)$ $|X| \leq 2|Y|$; $(ii)$ every triangle containing a vertex of $V_0$ has an edge in $X$; and $(iii)$ if $uv \in E(A)$ for some $A \in Y$, and $u, v \notin V_0$, then $uv \in X$. When $V_0$, $X$, and $Y$ satisfy the definition above, we say that $V_0$ is *reducible using* $X$ and $Y$. When $G$ has no reducible set, we say that $G$ is *irreducible*. The following lemma comes naturally.

**Lemma 3** ([Puleo 2015, Lemma 2.2])**.** *Let $G$ be a graph and $V_0 \subseteq V(G)$ be reducible using $X$ and $Y$. Let $G' = (G - X) - V_0$. If $\tau(G') \leq 2\nu(G')$, then $\tau(G) \leq 2\nu(G)$.*

We say that a graph $G$ is *robust* if, for every $v \in V(G)$, every component of $G[N(v)]$ has order at least 5. The following lemma (see [Puleo 2015, Lemma 2.7]) is an important tool in our proof. In what follows, the closed neighborhood $N(u) \cup \{u\}$ of a vertex $u \in V(G)$ is denoted by $N[u]$ and $\Delta(G)$ is the maximum degree in $G$.

**Lemma 4.** *If $G$ is an irreducible robust graph and $x$, $y \in V(G)$, then the following holds.*

(a) *if* $d(x) \leq 6$*, then* $\Delta(\overline{G[N(x)]}) \leq 1$ *and* $|E(\overline{G[N(x)]})| \neq 2$*;*
(b) *if* $d(x) \leq 6$ *and* $d(y) \leq 6$ *then* $xy \notin E(G)$*;*
(c) *if* $d(x) = 7$ *and* $d(y) = 6$ *then* $N[y] \not\subseteq N[x]$*;*
(d) *if* $d(x) \leq 8$ *and* $d(y) = 5$*, then* $N[y] \not\subseteq N[x]$.

In this paper we extend the result above to the following lemma.

**Lemma 5.** *If* $G$ *is an irreducible robust graph and* $x, y \in V(G)$ *are such that* $xy \notin E(G)$*,* $d(x), d(y) \leq 6$*, and* $|N(x) \cup N(y)| \leq 7$*, then* $d(x) = d(y) = 5$*,* $|N(x) \cap N(y)| = 3$*, and* $G[N(x)], G[N(y)] \simeq K_5$.

The following theorem is the main result of this paper.

**Theorem 6.** *If* $G$ *is a graph with treewidth at most* 6*, then* $\tau(G) \leq 2\nu(G)$.

*Proof.* Suppose, for a contradiction, that the statement does not hold, and let $G$ be a minimal counterexample. It is not hard to check that $|V(G)| \geq 8$. We claim that $G$ is irreducible. Indeed, suppose that there is a set $V_0 \subseteq V(G)$ that is reducible using $X$ and $Y$, and let $G' = (G - X) - V_0$. Since $G' \subseteq G$, we have $tw(G') \leq 6$, and by the minimality of $G$, we have $\tau(G') \leq 2\nu(G')$. Thus, by Lemma 4, $\tau(G) \leq 2\nu(G)$, a contradiction. It is not hard to check that $G$ is also robust, and hence has minimum degree at least 5.

By Proposition 1, $G$ has a rooted tree decomposition $(T, \mathcal{V}, r)$. Since $|V(G)| \geq 8$, we have $|V(T)| > 1$, and hence there is a node $t \in V(T)$ with $h(t) = 1$. Suppose that $S(t) = \{t'\}$, and $V_t = \{v_t, v_1, v_2, v_3, v_4, v_5, v_6\}$, where $v_t$ is the representative of $t$ if $t \neq r$. Since $(T, \mathcal{V}, r)$ is a rooted tree decomposition of $G$, we may assume that $V_{t'} = \{v_t, v_1, v_2, v_3, v_4, v_5, x\}$, with $x \neq v_6$. By Proposition 2, $d(x) \leq 6$, which implies, that $d(x) \in \{5, 6\}$. Also, by Proposition 2 applied to $G - x$ and $t$, we have that $N_{G-x}(v_t) \subseteq V_t$. Thus $d(v_t) = |N(v_t)| \leq |N_{G-x}(v_t)| + 1 \leq |V_t| = 7$.

Suppose that $d(v_t) = 7$. Now $N[x] \not\subseteq N[v_t]$ since $d(x) \in \{5, 6\}$, either by Lemma 4(c) or by Lemma 4(d). But, since $d(v_t) = 7$, note that $v_t$ is adjacent to $x$ and to every vertex in $V_t \setminus \{v_t\}$, hence $N[x] \subseteq N[v_t]$, a contradiction. So we may assume that $d(v_t) \leq 6$, and hence $v_t x \notin E(G)$ by Lemma 4(b). Since $d(v_t) \geq 5$, there are at least four neighbors of $v_t$ in $\{v_1, v_2, v_3, v_4, v_5\}$, say $v_1, v_2, v_4, v_5$. Now, since $v_t x \notin E(G)$, we have $d(x) = 5$. Thus, by Lemma 4(a), we have that $|E(G[N(x)])| \in \{9, 10\}$. If $|E(G[N(x)])| = 10$, then put

$$X = (\{v_t v_6\} \cap E(G)) \cup E(G[\{v_1, v_2, v_3, v_4, v_5\}]) \text{ and } Y = \{v_1 v_2 v_3, v_3 v_4 v_5, v_1 v_4 x, v_2 v_5 x, v_1 v_5 v_t, v_2 v_4 v_t\}.$$

Note that $|X| \leq 11 \leq 12 = 2|Y|$. It is not hard to check that $V_0 = \{v_t, x\}$ is reducible using $X$ and $Y$, a contradiction. So we may assume that $|E(G[N(x)])| = 9$. Note that the missing edge $e \notin E(G[N(v)])$ must be incident to a neighbor, say $v_5$, of $v_t$. Thus we may assume that $e \in \{v_3 v_5, v_4 v_5\}$. Put

$$X = (\{v_t v_6\} \cap E(G)) \cup E(G[\{v_1, v_2, v_3, v_4, v_5\}]) \text{ and } Y = \{v_1 v_2 v_3, v_1 v_5 x, v_3 v_4 x, v_1 v_4 v_t, v_2 v_5 v_t\}.$$

Note that $|X| \leq 10 = 2|Y|$. It is not hard to check that $V_0 = \{v_t, x\}$ is reducible using $X$ and $Y$, a contradiction.

Therefore we may assume that $|S(t)| > 1$. First, suppose that $|S(t)| \geq 3$, and let $t_1, t_2, t_3 \in S(t)$, and vertices $x, y, z$ be the representatives of $t_1, t_2, t_3$, respectively.

Note that, by Proposition 2, $xy, xz, yz \notin E(G)$ and $|N(x) \cup N(y)|$, $|N(x) \cup N(z)|$, and $|N(y) \cup N(z)| \leq |V_t| = 7$. Thus, by Lemma 5, $d(x) = d(y) = d(z) = 5$, $|N(x) \cap N(y)|$=$|N(x) \cap N(z)|$=$|N(y) \cap N(z)| = 3$, and $G[N(x)], G[N(y)], G[N(z)] \simeq K_5$. Assume without loss of generality that $N(x) = \{v_1, v_2, v_3, v_4, v_5\}$. Since $|N(x) \cap N(y)| = 3$, we may assume, without loss of generality, that $N(y) = \{v_3, v_4, v_5, v_6, v_7\}$. It is not hard to check that, since $|N(x) \cap N(z)| = |N(y) \cap N(z)| = 3$ and $|N(z)| = 5$, then $N(z)$ contains exactly one vertex in $N(x) \cap N(y)$. So we may assume, without loss of generality, that $N(z) = \{v_1, v_2, v_4, v_6, v_7\}$. Let $H = G[\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}]$ and note that every pair of vertices of $H$ is contained in at least one $N(v)$ for $v \in \{x, y, z\}$. Thus, since $G[N(v)]$ is isomorphic to $K_5$ for $v \in \{x, y, z\}$, $H$ is isomorphic to $K_7$. Let $X = E(H)$ and note that $|X| = 21$. Put

$$\begin{aligned} Y_1 = & \ \{xv_1v_3, xv_2v_5, yv_3v_4, yv_5v_6, zv_1v_6, zv_2v_4\}; \\ Y_2 = & \ \{v_1v_2v_7, v_2v_3v_6, v_4v_6v_7, v_3v_5v_7, v_1v_4v_5\}, \end{aligned}$$

and note that if $Y = Y_1 \cup Y_2$, then $|Y| = 11$ and $|X| \leq 2|Y|$. Hence $V_0 = \{x, y, z\}$ is reducible using $X$ and $Y$, a contradiction. We conclude that $|S(t)| \leq 2$.

If $t \neq r$, we let $w$ be the representative of $t$, otherwise we let $w$ be an arbitrary vertex of $V_t$. Let $t_1, t_2 \in S(t)$ and let $x, y$ be the representatives of $t_1, t_2$, respectively. Again, by Lemma 5, we have $d(x) = d(y) = 5$, $|N(x) \cap N(y)| = 3$, and $G[N(x)], G[N(y)] \simeq K_5$. Note that $t$ is a leaf of $(T', \mathcal{V}', r)$, where $T' = T - t_1 - t_2$ and $\mathcal{V}' = \mathcal{V} \setminus \{V_{t_1}, V_{t_2}\}$, hence $d_{G-x-y}(w) \leq 6$. Thus, we have $d_G(w) \leq 8$. Note that $w \in N(x) \cup N(y)$ and assume, without loss of generality, that $w \in N(x)$. Since $G[N(x)]$ is a complete graph, we have $N[x] \subseteq N[w]$, a contradiction to Lemma 4(d). This concludes the proof. □

## 3. Concluding remarks

This work has benefited greatly from [Puleo 2015]. Nevertheless, there were still gaps that we were able to explore. As we can see, the tree decomposition, specially under bounded treewidth, provides a suitable structure for problems of this nature. We believe that these techniques may be further improved by studying the behaviour of nodes from the tree decomposition with different heights.

## References


Bodlaender, H. (1998). A partial $k$-arboretum of graphs with bounded treewidth. Theoretical Computer Science, 209(1):1–45.

Gross, J. (2014). Embeddings of graphs of fixed treewidth and bounded degree. ARS Mathematica Contemporanea, 7:379–403.

Haxell, P. E. and Kohayakawa, Y. (1998). Packing and covering triangles in tripartite graphs. Graphs and Combinatorics, 14(1):1–10.

Puleo, G. (2015). Tuza's conjecture for graphs with maximum average degree less than 7. European Journal of Combinatorics, 49:134–152.

Tuza, Z. (1981). Conjecture in: finite and infinite sets. In Proc. Colloq. Math. Soc. J. Bolyai (Eger, Hungary, 1981), volume 37, page 888.

Tuza, Z. (1990). A conjecture on triangles of graphs. Graphs and Combinatorics, 6(4):373–380.