α-diperfect digraphs *

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Abstract. *Let D be a digraph. A* path partition *$\mathcal{P}$ of D is a collection of paths such that $\{V(P)\colon P \in \mathcal{P}\}$ is a partition of $V(D)$. We say D is α-*diperfect *if for every maximum stable set S of D there exists a path partition $\mathcal{P}$ of D such that $|S \cap V(P)| = 1$ for all $P \in \mathcal{P}$ and this property holds for every induced subdigraph of D. A digraph C is an* anti-directed odd cycle *if (i) the underlying graph of C is a cycle $x_1x_2 \cdots x_{2k+1}x_1$, where $k \geq 2$, (ii) the longest path in C has length 2, and (iii) each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \ldots, x_{2k}$ is either a source or a sink. Berge (1982) conjectured that a digraph D is α-diperfect if, and only if, D contains no induced anti-directed odd cycle. In this work, we verify this conjecture for digraphs whose underlying graph is series-parallel and for in-semicomplete digraphs.*

1. Introduction

All digraphs considered in this text are finite and contain neither loops nor parallel arcs (but they may contain cycles of length 2). For terminology not defined here, we refer the reader to [Bondy and Murty 2008]. Given a digraph D, we denote its vertex set by $V(D)$ and its arc set by $A(D)$. A pair of vertices $u, v \in V(D)$ is *adjacent* in D if $\{uv, vu\} \cap A(D) \neq \emptyset$. A *stable set* of a digraph D is a set $S \subseteq V(D)$ such that no pair of distinct vertices $u, v \in S$ is adjacent in D. The *stability number* of D, denoted by $\alpha(D)$, is the size of the largest stable set in D. A *path* P in D is a sequence $v_0v_1 \cdots v_\ell$ of distinct vertices of D such that $v_iv_{i+1} \in A(D)$ for $i = 0, \ldots, \ell - 1$. A *path partition* $\mathcal{P}$ of D is a collection of paths such that $\{V(P)\colon P \in \mathcal{P}\}$ is a partition of $V(D)$.

In 1960, Gallai and Milgram [Gallai and Milgram 1960] showed that the size of a minimum path partition of a digraph D is at most the stability number of D. Although various proofs of this result were known at the early 80's, no proof implied the existence of a maximum stable set S and a path partition $\mathcal{P}$ such that $|S \cap V(P)| = 1$ for every $P \in \mathcal{P}$ (later, Meyniel [Meyniel 1989] showed that there exist digraphs where such stable set and partition does not exist). Thinking about this matter, Berge [Berge 1982] proposed the class of α-diperfect digraphs. Given a digraph D and a stable set S of D, an *S-path partition* of D is a path partition $\mathcal{P}$ such that $|S \cap V(P)| = 1$ for all $P \in \mathcal{P}$. We say that D satisfies the *α-property* if, for every maximum stable set S of D, there exists an S-path partition of D, and we say that D is *α-diperfect* if every induced subdigraph of D satisfies the α-property.

*M. Sambinelli was partially supported by CNPq (Proc. 141216/2016-6) and FAPESP (Proc. 2017/23623-4). O. Lee was partially supported by CNPq (Proc. 311373/2015-1 and 425340/2016-3) and FAPESP (Proc. 2015/11937-9). E-mails: msambinelli@ic.unicamp.br (M. Sambinelli), candida@ufscar.br (C. N. da Silva), lee@ic.unicamp.br (O. Lee).

Given a digraph D, we denote its underlying graph by $U(D)$ (in this text we always consider that the underlying graph is simple). A digraph C is an *anti-directed odd cycle* if (i) $U(C) = x_1x_2 \cdots x_{2k+1}x_1$ is a cycle, where $k \geq 2$, (ii) the longest path in C has length 2, and (iii) each of the vertices $x_1, x_2, x_3, x_4, x_6, x_8, \ldots, x_{2k}$ is either a source or a sink. Berge [Berge 1982] showed that anti-directed odd cycles do not satisfy the α-property, and hence are not α-diperfect, which led him to conjecture the following characterization for α-diperfect digraphs. Note that it is strikingly similar to Berge's conjecture on perfect graphs – nowadays known as the Strong Perfect Graph Theorem (see Theorem 1 [Chudnovsky et al. 2006]).

Conjecture 1 (Berge, 1982) *A digraph D is α-diperfect if, and only if, D contains no induced anti-directed odd cycle.*

Theorem 1 (Chudnovsky, Robertson, Seymour, and Thomas, 2006) *A graph G is perfect if, and only if, neither G nor its complement contain an induced odd cycle of order at least* 5.

By showing that anti-directed odd cycles do not satisfy the α-property, Berge ended up showing the necessity of Conjecture 1. So the open problem in this conjecture is to verify its sufficiency. Still in his seminal paper, Berge [Berge 1982] showed that digraphs whose underlying graph is perfect and symmetric digraphs are both α-diperfect. To the best of our knowledge these are the only particular cases verified for Conjecture 1. The lack of results for this conjecture and the complexity of the proof of Theorem 1 as well as the time it took to prove it, seem to indicate that this is a very challenging problem.

A graph G is *series-parallel* if it can be obtained from the null graph by applying the following operations repeatedly: (i) adding a vertex v with degree at most one; (ii) adding a loop; (iii) adding a parallel edge; (iv) subdividing an edge. A *clique* of a digraph D is a set $S \subseteq V(D)$ such that every pair of vertices in S are adjacent in D. A digraph D is *semicomplete* if $V(D)$ is a clique, and D is *in-semicomplete* if, for every vertex $v \in V(D)$, the set $\{u\colon uv \in A(D)\}$ is a clique. Note that out-trees, cycles, and semicomplete digraphs are all in-semicomplete digraphs.

Series-parallel graphs are a common start point towards verifying graph theoretical conjectures [Chen et al. 2017, Juvan et al. 1999, Merker 2015] and in-semicomplete digraphs have been well studied in literature [Guo and Volkmann 1994, Bang-Jensen et al. 1997] and have been used to confirm open conjectures on digraphs [Bang-Jensen et al. 2006, Galeana-Sánchez and Gómez 2008]. The contributions of this work are the following theorems.

Theorem 2 *Let D be a digraph containing no induced anti-directed odd cycle. If $U(D)$ is series-parallel, then D is α-diperfect.*

Theorem 3 *If D is an in-semicomplete digraph, then D is α-diperfect.*

2. Outline of the proofs

The proofs of Theorems 2 and 3 use the following auxiliary results.

Lemma 4 *If a digraph D can be partitioned into k induced subdigraphs, say H_1, $H_2, \ldots, H_k$, such that $k \geq 2$, every H_i satisfies the α-property, and $\alpha(D) = \sum_{i=1}^{k} \alpha(H_i)$, then D satisfies the α-property.*

Proof: Let S be a maximum stable set of D and let $S_i = S \cap V(H_i)$ for $i = 1, 2, \ldots, k$. Thus,

$$\alpha(D) = |S| = \sum_{i=1}^{k} |S_i| \leq \sum_{i=1}^{k} \alpha(H_i) = \alpha(D).$$

Hence, S_i is a maximum stable set of H_i, and since the latter satisfies the α-property, there exists an (S_i)-path partition $\mathcal{P}_i$ of H_i, for $i = 1, \ldots, k$. Therefore, $\mathcal{P} = \bigcup_{i=1}^{k} \mathcal{P}_i$ is an S-path partition of D. Since S is an arbitrary maximum stable set of D, the result follows. $\square$

Lemma 5 *If B is a clique cut of a digraph D, then D can be partitioned into two proper induced subdigraphs D_1 and D_2 such that $\alpha(D) = \alpha(D_1) + \alpha(D_2)$. Moreover, if uv is an arc of D such that $u \in V(D_1)$ and $v \in V(D_2)$, then $\{u, v\} \cap B \neq \emptyset$.*

Lemma 6 *Let D be a digraph. If $U(D)$ contains a proper induced cycle containing at most two vertices with degree greater than two, then D can be partitioned into two proper induced subdigraphs D_1 and D_2 such that $\alpha(D) = \alpha(D_1) + \alpha(D_2)$.*

2.1. Outline of the proof of Theorem 2

Towards a contradiction, suppose that the result does not hold, and let D be a counterexample with the smallest number of vertices. It is not hard to check that D has order at least 3. Moreover, since every subgraph of a series-parallel graph is also a series-parallel graph, we have, by the minimality of D, that every proper induced subdigraph of D satisfies the α-property, which means that D does not, since it is a counterexample. Then we show that D can be partitioned into two proper induced subdigraphs D_1 and D_2 such that $\alpha(D) = \alpha(D_1) + \alpha(D_2)$. We prove this result as follows. If D has a cut vertex, then the result follows by Lemma 5. Otherwise, D has no cut vertex, and hence $U(D)$ is 2-connected. We show that $U(D)$ contains an induced cycle C containing at most two vertices with degree greater than 2. If $U(D) = C$, then we show that D satisfies the α-property, a contradiction. Otherwise, C is a proper induced subgraph, and hence the result follows by Lemma 6. Therefore, there exists such partition of D and, by Lemma 4, D satisfies the α-property, a contradiction.

2.2. Outline of the proof of Theorem 3

Towards a contradiction, suppose that the result does not hold and let D be a counterexample with the smallest number of vertices. It is not hard to check that D has order at least 3. Moreover, since every induced subgraph of an in-semicomplete digraph is also an in-semicomplete digraph, we have, by the minimality of D, that every proper induced subdigraph of D satisfies the α-property, which means that D does not, since it is a counterexample. If D is disconnected, then $\alpha(D) = \alpha(C) + \alpha(D - V(C))$, where C is a component of D, and hence, by Lemma 4, D satisfies the α-property, a contradiction. Therefore, we may assume that D is connected. If D is strong, then we use the following result provided in [Bang-Jensen et al. 1993] to show that D satisfies the α-property, a contradiction.

Theorem 7 (Bang-Jensen, Huang, and Prisner, 1993) *An in-semicomplete digraph D of order at least 2 is hamiltonian if, and only if, D is strong.*

Since D is a connected non-strong digraph, there exists a strong component X such that no arc in D is leaving X. Let $Y = \{v \in V(D) \setminus V(X) \colon vu \in A(D) \text{ and } u \in V(X)\}$. By a result of Bang-Jensen and Gutin [Bang-Jensen and Gutin 1998], we have $yx \in A(D)$ for every $y \in Y$ and $x \in V(X)$. Thus, since D is in-semicomplete, Y is a clique. If Y is a cut of D, then, by Lemma 5, D can be partitioned into two proper induced subdigraphs D_1 and D_2 such that $\alpha(D) = \alpha(D_1) + \alpha(D_2)$, and hence, by Lemma 4, D satisfies the α-property, a contradiction. Thus, we may assume that Y is not a vertex cut, and hence $V(D) = Y \cup V(X)$. We note that $u \in Y$ is a vertex adjacent to every vertex of D and prove that if $D - u$ satisfies the α-property, then D also satisfies the α-property, a contradiction.

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