Vertex partition problems in digraphs *

M. Sambinelli[1], C. N. Lintzmayer[2], C. N. da Silva[3], O. Lee[4]

[1]Institute of Mathematics and Statistics – University of São Paulo – Brazil

[2]Center for Mathematics, Computation and Cognition – Federal University of ABC – Brazil

[3]Department of Computing – Federal University of São Carlos – Brazil

[4]Institute of Computing – University of Campinas – Brazil

Abstract. *Let D be a digraph and k be a positive integer. Linial (1981) conjectured that the k-norm of a k-minimum path partition of a digraph D is at most $\max\{\sum_{C \in \mathcal{C}} |C| \colon \mathcal{C}$ is a partial k-coloring of $D\}$. Berge (1982) conjectured that every k-minimum path partition contains a partial k-coloring orthogonal to it. It is well known that Berge's Conjecture implies Linial's Conjecture. In this work, we verify Berge's Conjecture, and consequently Linial's Conjecture, for locally in-semicomplete digraphs and k-minimum path partitions containing only two paths. Moreover, we verify a conjecture related to Berge's and Linial's Conjectures for locally in-semicomplete digraphs.*

1. Introduction

Given a digraph D, we denote its vertex set by $V(D)$ and its arc set by $A(D)$. A *path partition* $\mathcal{P}$ of a digraph D is a collection of paths such that $\{V(P) \colon P \in \mathcal{P}\}$ is a partition of $V(D)$. Given a positive integer k, the *k-norm* of $\mathcal{P}$, denoted by $|\mathcal{P}|_k$, is $\sum_{P \in \mathcal{P}} \min\{|V(P)|, k\}$. We say that $\mathcal{P}$ is *k-minimum* if $|\mathcal{P}|_k \leq |\mathcal{P}'|_k$ for every path partition $\mathcal{P}'$ of D, and we denote by $\pi_k(D)$ the k-norm of a k-minimum path partition of D. A *partial k-coloring* $\mathcal{C}$ of D is a collection of stable sets such that $\mathcal{C}$ is a packing of $V(D)$ and $|\mathcal{C}| \leq k$. Let $\alpha_k(D) = \max\{\sum_{C \in \mathcal{C}} |C| \colon \mathcal{C}$ is a partial k-coloring of $D\}$. In 1981, Linial [Linial 1981] proposed Conjecture 1 which extends the classical result of Gallai and Milgram (see [Hartman 2006]) that says that the size of a minimum path partition of a digraph is at most its stability number.

Conjecture 1 (Linial, 1981) *If D is a digraph and $k \in \mathbb{Z}^+$, then $\pi_k(D) \leq \alpha_k(D)$.*

A path partition $\mathcal{P}$ of a digraph D and a partial k-coloring $\mathcal{C}$ of D are *orthogonal* if each $P \in \mathcal{P}$ meets $\min\{|V(P)|, k\}$ stable sets in $\mathcal{C}$. In an attempt to unify the proof of Gallai and Milgram's result with another classical result in graph theory (see [Berge 1997]), Berge [Berge 1982] proposed Conjecture 2. It is known that this conjecture implies Conjecture 1 (see [Hartman 2006]).

Conjecture 2 (Berge, 1982) *Let D be a digraph and $k \in \mathbb{Z}^+$. If $\mathcal{P}$ is a k-minimum path partition of D, then there exists a partial k-coloring of D orthogonal to $\mathcal{P}$.*

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Conjectures 1 and 2 remain open, but they were verified for some particular cases. Conjecture 2, and hence Conjecture 1, was verified for $k = 1$ [Linial 1978], $k = 2$ [Berger and Hartman 2008], $k \geq \lambda - 3$ (where λ is the order of a longest path) [Herskovics 2016], when all the paths from the k-minimum path partition have order at most k [Berge 1982] or at least k [Aharoni and Hartman 1993], acyclic digraphs [Aharoni et al. 1985], and bipartite digraphs [Berge 1982]. Moreover, Conjecture 1 was verified for a superclass of split digraphs [Sambinelli et al. 2017].

Let D be a digraph and k be a positive integer. A *path k-pack* $\mathcal{P}$ of D is a collection of paths such that $\{V(P)\colon P \in \mathcal{P}\}$ is a packing of $V(D)$ and $|\mathcal{P}| \leq k$. The *weight* of $\mathcal{P}$, denoted by $||\mathcal{P}||$, is $\sum_{P\in\mathcal{P}} |V(P)|$, and we say that $\mathcal{P}$ is *maximum* if $||\mathcal{P}|| \geq ||\mathcal{P}'||$ for every path k-pack $\mathcal{P}'$ of D. A *coloring* $\mathcal{C}$ of D is a collection of stable sets such that $\mathcal{C}$ is a partition of $V(D)$. A path k-pack $\mathcal{P}$ and a coloring $\mathcal{C}$ are *orthogonal* if each stable set $C \in \mathcal{C}$ meets $\min\{|C|, k\}$ paths in $\mathcal{P}$. As a generalization for a conjecture related to Conjecture 1, Aharoni, Hartman, and Hoffman [Aharoni et al. 1985] proposed Conjecture 3 – to understand the relationship among these conjectures, see [Hartman 2006]. Conjecture 3 was verified for $k = 1$ [Gallai 1968], when the maximum path k-pack has at least one trivial path [Hartman et al. 1994], bipartite digraphs [Hartman et al. 1994], and acyclic digraphs [Aharoni et al. 1985].

Conjecture 3 (Aharoni, Hartman, and Hoffman, 1985) *Let D be a digraph and $k \in \mathbb{Z}^+$. If $\mathcal{P}$ is a maximum path k-pack of D, then there is a coloring of D orthogonal to $\mathcal{P}$.*

A digraph D is *semicomplete* if $V(D)$ is a clique, and it is *in-semicomplete* if, for every vertex $v \in V(D)$, the set $\{u\colon uv \in A(D)\}$ is a clique. Note that out-trees, cycles, and semicomplete digraphs are all in-semicomplete digraphs. In-semicomplete digraphs have been well studied in literature [Guo and Volkmann 1994, Bang-Jensen et al. 1997] and have been used as a particular case to confirm open conjectures on digraphs [Bang-Jensen et al. 2006, Galeana-Sánchez and Gómez 2008]. The contributions of this work are the following theorems.

Theorem 1 *If $\mathcal{P}$ is a k-minimal path partition of an in-semicomplete digraph D, then there exists a partial k-coloring of D orthogonal to $\mathcal{P}$.*

Theorem 2 *If $\mathcal{P}$ is a maximum path k-pack of an in-semicomplete digraph D, then there exists a coloring of D orthogonal to $\mathcal{P}$.*

Theorem 3 *Let D be a digraph and let k be a positive integer. If $\mathcal{P} = \{P_1, P_2\}$ is a k-minimum path partition of D, then there exists a partial k-coloring orthogonal to $\mathcal{P}$.*

Theorems 1 and 3 confirm, respectively, Conjecture 2 for in-semicomplete digraphs and for k-minimum path partitions containing only two paths. Theorem 2 confirms Conjecture 3 for in-semicomplete digraphs.

2. Brief outline of the proofs of Theorems 1 and 2

Our proofs for both theorems use induction and follow similar ideas. One important structure that they rely on is the following characterization of in-semicomplete digraphs [Bang-Jensen and Gutin 2009]. Given a collection of paths $\mathcal{P}$, the set of terminal vertices of paths in $\mathcal{P}$ is denote by $\operatorname{ter}(\mathcal{P})$, i.e., $\operatorname{ter}(\mathcal{P}) = \{v_\ell\colon v_1 v_2 \cdots v_\ell = P \in \mathcal{P}\}$.

Theorem 4 (Bang-Jensen and Gutin, 2009) *A digraph D is in-semicomplete if, and only if, for every vertex v and every pair of internally vertex-disjoint paths P and Q ending at v, there exists a path R ending at v such that $V(R) = V(P) \cup V(Q)$.*

2.1. Theorem 1

Let D be an in-semicomplete digraph, let k be a positive integer, and let $\mathcal{P}$ be a path partition of D. Our proof consists in showing that (i) there exists a partial k-coloring of D orthogonal to $\mathcal{P}$; or (ii) there exists a path partition $\mathcal{B}$ of D such that $|\mathcal{B}|_k < |\mathcal{P}|_k$ and $\mathrm{ter}(\mathcal{B}) \subseteq \mathrm{ter}(\mathcal{P})$. We start our proof by showing that if (ii) does not hold, then there exists a path partition $\mathcal{Q}$ such that $|\mathcal{Q}|_k = |\mathcal{P}|_k$, $\mathrm{ter}(\mathcal{Q}) \subseteq \mathrm{ter}(\mathcal{P})$, $\mathrm{ter}(\mathcal{Q})$ is stable, and every partial k-coloring orthogonal to $\mathcal{Q}$ is also orthogonal to $\mathcal{P}$. Our proof for such result follows by induction on the number of paths in $\mathcal{P}$ with order smaller than k. This reduces the problem of proving the result for $\mathcal{P}$ to the problem of proving it for $\mathcal{Q}$. The remaining proof follows by induction of k. If $k = 1$, then the stable set $\mathrm{ter}(\mathcal{Q})$ is a partial 1-coloring orthogonal to $\mathcal{Q}$, and to $\mathcal{P}$, and the result follows. Otherwise, $k > 1$ and let $D' = D - \mathrm{ter}(\mathcal{Q})$ and $\mathcal{Q}' = \{Q_1 \colon Q_1 u = Q \in \mathcal{Q}\}$. Note that D' is an in-semicomplete digraph and that $\mathcal{Q}'$ is a path partition of D'. By the induction hypothesis applied to D', $\mathcal{Q}'$, and $k-1$, there exists (a) a partial $(k-1)$-coloring $\mathcal{C}$ of D' orthogonal to $\mathcal{Q}'$, or (b) a path partition $\mathcal{R}'$ of D' such that $|\mathcal{R}'|_{k-1} < |\mathcal{Q}'|_{k-1}$ and $\mathrm{ter}(\mathcal{R}') \subseteq \mathrm{ter}(\mathcal{Q}')$. If (a) holds, then $\mathcal{C} \cup \{\mathrm{ter}(\mathcal{Q})\}$ is a partial k-coloring orthogonal to $\mathcal{Q}$ and (i) holds. So we assume that (b) holds and, with the help of Theorem 4, we show how to build a path partition of D satisfying (ii) from $\mathcal{Q}'$.

2.2. Theorem 2

Let D be an in-semicomplete digraph, let k be a positive integer, and let $\mathcal{P}$ be a path k-pack of D. Our proof consists in showing that (i) there exists a coloring of D orthogonal to $\mathcal{P}$, or (ii) there exists a path k-pack $\mathcal{B}$ of D such that $||\mathcal{B}|| = ||\mathcal{P}|| + 1$ and $\mathrm{ter}(\mathcal{B}) \subseteq \mathrm{ter}(\mathcal{P}) \cup \overline{V}_{\mathcal{P}}$, where $\overline{V}_{\mathcal{P}} = V(D) \setminus \cup_{P \in \mathcal{P}} V(P)$. Our proof follows by induction on $|\overline{V}_{\mathcal{P}}|$. If $\overline{V}_{\mathcal{P}} = \emptyset$, then the coloring $\{\{v\} \colon v \in V(D)\}$ is orthogonal to $\mathcal{P}$ and (i) holds. Thus, we may assume $\overline{V}_{\mathcal{P}} \neq \emptyset$. Let w be a vertex in $\overline{V}_{\mathcal{P}}$ and let $\mathcal{Q} = \mathcal{P} \cup \{w\}$. If $|\mathcal{Q}| \leq k$, then $\mathcal{B} = \mathcal{Q}$ satisfies (ii) and the result follows. Thus we may assume that $|\mathcal{Q}| > k$, and, in this case, $|\mathcal{Q}| = k+1$ and $|\mathcal{P}| = k$, since $|\mathcal{P}| \leq k$. Let $S \subseteq \overline{V}_{\mathcal{P}}$ be a maximum stable set in $D[\overline{V}_{\mathcal{P}}]$ and let $Z = \mathrm{ter}(\mathcal{P}) \cup S$. We can prove that Z is a stable set of D. Let $D' = D - Z$, and let $\mathcal{P}' = \{P' \colon P'u = P \in \mathcal{P}\}$. Note that D' is an in-semicomplete digraph, $\mathcal{P}'$ is a path k-pack of D', $||\mathcal{P}'|| = ||\mathcal{P}|| - k$, and $\overline{V}'_{\mathcal{P}} \subset \overline{V}_{\mathcal{P}}$. By the induction hypothesis applied to D' and $\mathcal{P}'$ we have (a) there exists a coloring $\mathcal{C}$ of D' orthogonal to $\mathcal{P}'$, or (b) there exists a path k-pack $\mathcal{Q}'$ of D' such that $||\mathcal{Q}'|| = ||\mathcal{P}'|| + 1$ and $\mathrm{ter}(\mathcal{Q}') \subseteq \mathrm{ter}(\mathcal{P}') \cup \overline{V}'_{\mathcal{P}}$. If (a) holds, then $\mathcal{C} \cup \{Z\}$ is a coloring of D orthogonal to $\mathcal{P}$ and (i) holds. So we assume that (b) holds and, with the help of Theorem 4 and Hall's theorem, we show how to extend the path k-pack $\mathcal{Q}'$ to a path k-pack of D that satisfies (ii).

3. Brief outline of the proof of Theorem 3

Let D be a digraph, let k be a positive integer, and let $\mathcal{P} = \{P_1, P_2\}$ be a k-minimum path partition of D. Let $P_1 = u_1 u_2 \cdots u_\ell$ and let $P_2 = v_1 v_2 \cdots v_r$. If either $\ell, r \leq k$ or $\ell, r \geq k$, then there exists a partial k-coloring orthogonal to $\mathcal{P}$ and the result follows [Berge 1982, Aharoni and Hartman 1993]. Thus we may assume, without loss of generality, that $\ell > k$ and $r < k$, and hence $|\mathcal{P}|_k = k + r$. We can prove that P_1 is a longest path of D. Let $X = V(P_2)$, $Y = V(P_1)$, and G be the $\{X, Y\}$ bipartite graph defined by $V(G) = X \cup Y$ and $E(G) = \{uv \colon u \in X, v \in Y$, and u and v are not adjacent in $D\}$. Using Hall's theorem, we can show that there exists a matching M in G covering the vertices of X, otherwise P_1 would not be a longest path in D. Given a vertex $v \in X$, we write $M(v)$ to denote the vertex in $Y = V(P_1)$ matched to v in M. By the construction of G, v and $M(v)$ are non-adjacent. Let $\mathcal{S}_1 = \{\{u, M(u)\}: u \in X\}$, and let $\mathcal{S}_2$ be a set of $k - r$ vertices of P_1 not matched by M. Thus, $\mathcal{S}_1 \cup \{\{u\}: u \in \mathcal{S}_2\}$ is a partial k-coloring orthogonal to $\mathcal{P}$.

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