Edge-colouring of triangle-free graphs with no proper majors

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Abstract. *The problem of determining the chromatic index of a graph, even when triangle-free, is* NP*-hard. However, the* Overfull Conjecture *implies that this problem can be solved in polynomial time for graphs with large maximum degree. In order to prove the conjecture, it is sufficient to show that all* non-subgraph-overfull *graphs with* $\Delta > n/3$ *are Class 1. A special case of non-subgraph-overfull graphs are the graphs with no proper majors. We show that all triangle-free graphs with no proper majors, regardless of maximum degree, are Class 1. We also provide a polynomial-time algorithm to colour such graphs.*

1. Introduction

Throughout this text, a *graph* is always a simple graph, which means undirected, loopless and with no multiple edges. We follow the usual definitions and notation for graph-theoretical related concepts. In particular, given a vertex u in a graph G, we denote the *set of neighbours* of u in G by $N_G(u)$ and the *degree* of u in G by $d_G(u) = |N_G(u)|$.

Let G be an n-order graph of maximum degree Δ. A k*-edge-colouring* of G is a function $\varphi\colon E(G) \to [1..k]$ such that adjacent edges have different images (or *colours*). The *chromatic index* of G, denoted by $\chi'(G)$, is the least k for which G is k-edge-colourable. A breakthrough by [Vizing 1964] uses a recolouring procedure in order to show that $\chi'(G)$ is either Δ or $\Delta + 1$. The graph G is said to be *Class 1* (*Class 2*) in the former (latter) case. Deciding if a graph is *Class 1* is an NP-complete problem [Holyer 1981], even when restricted to triangle-free graphs with $\Delta = 3$ [Koreas 1997].

We say that G is *overfull* if it has more than than $\Delta\lfloor n/2\rfloor$ edges, or, equivalently, if n is odd and $\sum_{u\in V(G)}(\Delta - d_G(u)) \leq \Delta - 2$ [Niessen 1994]. We say that G is *subgraph-overfull* (*SO*) if it has a Δ-*overfull* subgraph, i.e. an overfull subgraph H with $\Delta(H) = \Delta$. Being *SO* is clearly a sufficient condition for G to be *Class 2*. The *Overfull Conjecture* states that this condition is also necessary when $n > \Delta/3$ [Hilton and Johnson 1987]. This is particularly interesting because deciding if G is *SO* can be done in polynomial-time when $\Delta > n/3$ and in linear-time when $\Delta \geq n/2$ [Niessen 2001]. The conjecture holds for triangle-free graphs with $\Delta \geq n/2$ (which are all *Class 1*) [Zorzi and Zatesko 2016].

Following [Niessen 1994], we call a vertex of maximum degree in G a *major* of G. A *proper major* of G is a major x of G which satisfies

$$\sum_{u\in N_G(x)} (\Delta - d_G(u)) \leq \Delta - 2. \tag{1}$$

We remark that x being a proper major does not imply that $H := N_G(x) \cup \{x\}$ induces an overfull graph, because for that to happen we should have (1) holding even when replacing

G by H. A *non-proper major* is a major which is not proper. As overfull graphs do not have non-proper majors [Niessen 1994], no graph with no proper majors can be *SO*. We show that all triangle-free graphs with no proper majors are *Class 1*, even if $\Delta \leq n/3$.

2. The result

We use in our result (Theorem 3) a lemma from [Zorzi and Zatesko 2016] which is a variation of Vizing's recolouring procedure [Vizing 1964]. We restate this lemma (Lemma 1) in terms of two properties, defined below, about a sequence of distinct vertices $\sigma = v_0, \ldots, v_k$ of a graph G. For the statements of the properties and of the lemma, let $\varphi\colon E(G) \to \mathscr{C}$ be a t-edge-colouring of G and uv be a non-edge of G.

Property $\mathscr{P}_1$. *All of the following holds:*

1. $v_0 = v$ *and* $\{v_1, \ldots, v_k\} \subseteq N_G(u)$*;*
2. *for all* $i \in \{0, \ldots, k-1\}$*, either* v_i *misses a colour* α_i*, or* $i > 0$ *and some* $w_i \in N_G(v_i) \setminus (N_G(u) \cup \{v_0\})$ *misses* α_{i-1}*, in which case we define* $\alpha_i := \varphi(v_i w_i)$*;*
3. *for all* $i \in \{1, \ldots, k\}$*, we have* $\varphi(uv_i) = \alpha_{i-1}$*.*

In Property $\mathscr{P}_1$.2, when v_i does not actually miss a colour α_i, but has a neighbour w_i which misses α_{i-1}, we say that v_i misses the colour $\alpha_i := \varphi(v_i w_i)$ *virtually*.

Property $\mathscr{P}_2$. *One of the following holds:*

1. *there is a colour missing at both* v_k *and* u*;*
2. $k > 0$ *and the colour* α_j*, for some* $j < k$*, is missing at* v_k*;*
3. $k > 0$ *and there is a vertex* $w_k \in N_G(v_k) \setminus (N_G(u) \cup \{v_0\})$ *which misses* α_{k-1} *such that* $\varphi(v_k w_k)$ *is missing at* u *or is equal to* α_j *for some* $j < k$*.*

Lemma 1 [Zorzi and Zatesko 2016]. *If* u *misses a colour of* $\mathscr{C}$ *and there is a sequence of distinct vertices* $\sigma = v_0, \ldots, v_k$ *satisfying both Properties* $\mathscr{P}_1$ *and* $\mathscr{P}_2$*, then* $G + uv$ *is also* t*-edge-colourable.*

For any two colours α and β used in an edge-colouring of a graph G, we define $G[\alpha, \beta]$ as the subgraph of G induced by all the edges coloured with α or β. Notice that every component of $G[\alpha, \beta]$ is a path or an even cycle. Considering this, it is worth remarking the following about Lemma 1, which, by the way, can be used to prove that every graph with no major vertex belonging to an odd cycle is *Class 1*.

Observation 2. *For every sequence* $\sigma = v_0, \ldots, v_k$ *satisfying Property* $\mathscr{P}_1$*, we can assume without loss of generality:*

1. *Every colour* $\beta \in \mathscr{C}$ *missing at* u *is not missing at any* v_i *for* $i \in \{0, \ldots, k-1\}$*, since otherwise, by taking the least such* i*, the sequence* $v_0, \ldots, v_i$ *would already satisfy all the properties required in Lemma 1.*
2. *For all* $\beta \in \mathscr{C}$ *missing at* u *and all* $i \in \{0, \ldots, k-1\}$*, the vertices* u *and* v_i *belong to the same component* P *in* $G[\alpha_i, \beta]$*, since otherwise, by exchanging the colours of the component to which* u *belongs, the sequence* $v_0, \ldots, v_i$ *would itself satisfy all the properties required in Lemma 1. Note that* P *is a path,* u *is an outer vertex of* P *and, if* v_i *does not miss* α_i *virtually, the other outer vertex is* v_i*.*
3. *For all* $i \in \{1, \ldots, k-1\}$ *and all* $j \in \{0, \ldots, i-1\}$*,* α_i *is not missing at* v_j *nor is* $\varphi(v_j w_j)$ *for some* $w_j \in N_G(v_j) \setminus (N_G(u) \cup \{v_0\})$ *which misses* α_{j-1}*, since otherwise, by repeatedly taking such* i *and* j*, we would be able to choose* α_i *for the role of* α_j*, thus replacing* $\sigma = v_0, \ldots, v_k$ *by* $\sigma' = v_0, \ldots, v_j, v_{i+1}, \ldots, v_k$*.*

Now we are ready to present our result:

Theorem 3. *Every triangle-free graph with no proper major vertex is Class 1.*

Proof. Let G be a triangle-free graph with maximum degree Δ. We shall prove for all $F \subseteq E(G)$ that $G[F]$ is Δ-edge-colourable (which does not necessarily mean that $G[F]$ is *Class 1* when $\Delta(G[F]) < \Delta$). Since $G[\emptyset]$ is Δ-edge-colourable, assume $|F| > 0$ and, by induction, that $G[F']$ is Δ-edge-colourable for all $F' \subseteq E(G)$ such that $|F'| < |F|$.

Let uv be any edge in F and $H := G[F \setminus \{uv\}]$. By the induction hypothesis, H admits a Δ-edge-colouring $\varphi\colon E(H) \to \mathscr{C}$. Let $\sigma = v_0, \ldots, v_k$ be a maximal sequence satisfying Property $\mathscr{P}_1$ and all the properties discussed in Observation 2. The sequence σ must be non-empty, since both u and v have degree less than Δ in H and thence miss a colour of $\mathscr{C}$ each. If σ also satisfies Property $\mathscr{P}_2$, we can apply Lemma 1 to show that $G[F]$ is Δ-edge-colourable and the proof is concluded. In case σ fails to satisfy Property $\mathscr{P}_2$, we demonstrate how to obtain a new sequence satisfying the two properties.

By the maximality of σ, it is straightforward to verify that the only possible reason for σ not satisfying Property $\mathscr{P}_2$ would be if $d_H(v_k) = \Delta$ and no vertex in $N_H(v_k) \setminus (N_H(u) \cup \{v_0\})$ missed $\alpha_{k-1} = \varphi(uv_k)$. By the way, note that $N_H(v_k) \setminus (N_H(u) \cup \{v_0\}) = N_H(v_k)$, since H is triangle-free. We have the following two cases to investigate.

Case 1. There is some colour β missing at u and also at some $y \in N_H(v_k) \setminus \{u\}$.

By Observation 2 we know that u and v_{k-1} belong to the same component P of $H[\alpha_{k-1}, \beta]$, which is a path and have u as one of its outer vertices. We have two sub-cases.

1. If y does not belong to P, then by exchanging the colours of the component of y in $H[\alpha_{k-1}, \beta]$ (also a path), we obtain a vertex in $N_H(v_k)$ missing α_{k-1} as desired. Furthermore, this does not cause any v_i with $i \in \{0, \ldots, k-2\}$ to miss β because, by Observation 2, neither β nor α_{k-1} is missing at v_i.
2. If y is in P, it is the other outer vertex of P, so v_{k-1} is an inner vertex of P and thence misses α_{k-1} virtually. Then, we first uncolour the edge $v_{k-1}w_{k-1}$ (which was coloured with α_{k-1}). Now, u and y are not in the same component of $H[\alpha_{k-1}, \beta]$ any more, and we are back to the previous sub-case. According to [Zorzi and Zatesko 2016], after the recolouring procedure is fully performed, v_{k-1} will miss α_{k-2}, and we will be able to colour $v_{k-1}w_{k-1}$ with α_{k-2}. See Figure 1 on the next page.

Note that neither sub-case interferes in any v_i for $i \in \{0, \ldots, k-1\}$.

Case 2. No vertex in $N_H(v_k) \setminus \{u\}$ misses α_{k-1} or any colour missing at u.

First recall that $d_H(v_k) = d_G(v_k) = \Delta$, $d_H(u) < d_G(u)$ and, thus,

$$\sum_{y \in N_H(v_k)} (\Delta - d_H(y)) \geq \sum_{y \in N_G(v_k)} (\Delta - d_G(y)) + 1 \geq \Delta.$$

Hence, by the Pigeonhole Principle, there must be two vertices $y_1, y_2 \in N_H(v_k) \setminus \{u\}$ missing the same colour $\gamma \in \mathscr{C}$. As it is possible for only one among y_1 and y_2 to belong to the same component of $H[\beta, \gamma]$ as u, say y_1, exchanging the colours of the component to which y_2 belongs yields β missing at both u and y_2. It should be noticed that this operation may cause some v_i with $i \in \{0, \ldots, k-1\}$ to miss β. In this case $v_0, \ldots, v_i$ would satisfy the two properties and we would be done. Otherwise, we are back to Case 1.

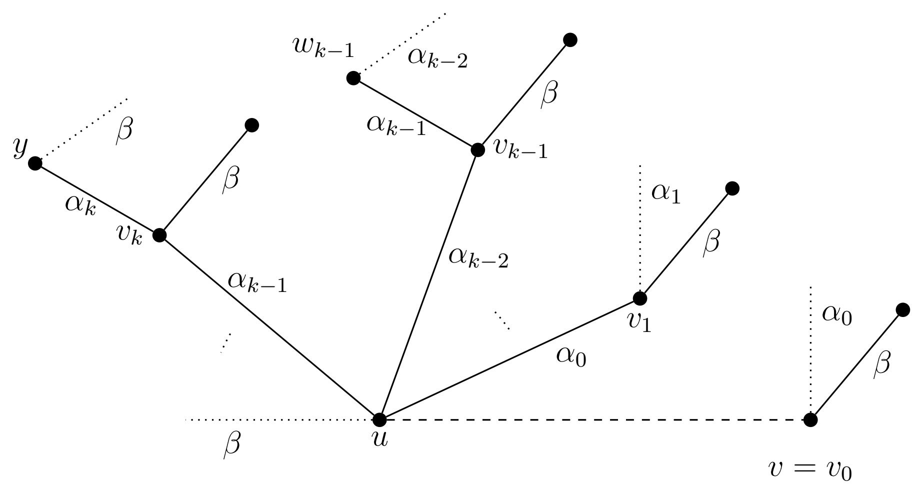

Figure 1. An illustration for Case 1.2. Here, the dashed line indicates the edge to be coloured and the dotted lines indicate the colours missing at the vertices.

Now, if we are not done yet, the sequence still satisfies Property $\mathscr{P}_1$ and there is some $w_k \in N_H(v_k)$ which misses α_{k-1} and defines $\alpha_k := \varphi(v_k w_k)$. So, if the sequence does not satisfy also Property $\mathscr{P}_2$, we can continue to construct it by appending to it the vertex v_{k+1} such that $\varphi(uv_{k+1}) = \alpha_k$, and so on, until the two properties are satisfied. In the process, we may append a major vertex x which does not have a neighbour missing $\varphi(ux)$, but this situation we have shown how to handle. $\square$

We finish by pointing out that the edge-colouring process described in the proof of Theorem 3 can be performed in polynomial time.

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