Hitting all longest cycles in a graph

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Abstract. *Let* $\mathrm{lct}(G)$ *be the minimum size of a set of vertices that contains at least one vertex in every longest cycle of a graph* G*. We show that* $\mathrm{lct}(G) = 1$ *if* G *is a 3-tree, and that* $\mathrm{lct}(G) \leq 2$ *if* G *is a 2-connected partial 3-tree.*

1. Introduction

It is known that, in every 2-connected graph, every pair of longest cycles intersect each other in at least two vertices. A natural question asks whether all longest cycles have a vertex in common in 2-connected graphs. (If the graph is not 2-connected, two longest cycles can be disjoint.) This has in general a negative answer, as the Petersen's graph shows. However, there are some graph classes for which this question has a positive answer, such as classes containing only Hamiltonian graphs [Thomas and Yu 1994, Tutte 1956], and dually chordal graphs [Jobson et al. 2016]. In this paper we show that the class of 3-trees also has a positive answer to this question. Observe that 3-trees are not dually chordal graphs, as they are not clique-Helly graphs, so they are not included in the class addressed by Jobson *et al.* [Jobson et al. 2016].

When one cannot find a vertex that is common to all longest cycles in a graph, it is interesting to look for a set of vertices such that every longest cycle has at least one vertex in that set. Such a set is called a longest cycle transversal, or just a transversal, and we look for small transversals, possibly of minimum size. The minimum size of a transversal is denoted by $\mathrm{lct}(G)$. When we cannot determine $\mathrm{lct}(G)$ exactly, it is interesting to search for a good upper bound for it. For every 2-connected graph G with n vertices, $\mathrm{lct}(G) \leq \lceil n/3 \rceil$ [Thomassen 1978]. This bound was later improved to $\mathrm{lct}(G) \leq \lceil \frac{n}{3} - \frac{n^{2/3}}{36} \rceil$ [Rautenbach and Sereni 2014]. Restating what was written at the end of the previous paragraph, here we show that $\mathrm{lct}(G) = 1$ when G is a 3-tree. We also show that, when G is a 2-connected partial 3-tree, $\mathrm{lct}(G) \leq 2$.

2. Preliminaries

Let C be a cycle in a graph G. Sometimes we will refer to C as the set of vertices of C. We denote by $|C|$ the length of C, that is, the number of edges in C. A triangle in G is a cycle of length three. Let S be a set of vertices of G. Given a cycle C with at least one vertex not in S, we say that S fences C if $C - S$ is contained in a single connected component of $G - S$, otherwise we say that C crosses S. For an integer t and a set S of vertices, we say that a cycle C t-touches S if C intersects S at exactly t vertices. Note that if C 1-touches S then C is fenced by S.

Lemma 1. *Let* G *be a 2-connected graph with a triangle* Δ*. Let* $\mathscr{C}$ *be the set of all longest cycles in* G *that cross* Δ*. At least two vertices of* Δ *are in all cycles of* $\mathscr{C}$*.*

Let X and Y be two subsets of $V(G)$. We denote by $\mathrm{Comp}_Y(X)$ the union of the vertex set of the connected components of $G - Y$ containing at least one vertex of $X \setminus Y$. If C is a cycle, then we may write $\mathrm{Comp}_Y(C)$ instead of $\mathrm{Comp}_Y(V(C))$. Also, if v is a vertex, we may write $\mathrm{Comp}_Y(v)$ instead of $\mathrm{Comp}_Y(\{v\})$. The next lemma is an adaptation of Lemma 6 of de Rezende *et al.* [De Rezende et al. 2013].

Lemma 2. *Let G be a 2-connected graph with a triangle Δ. If there are longest cycles C' and C'' such that $\mathrm{Comp}_\Delta(C') \cap \mathrm{Comp}_\Delta(C'') = \emptyset$, then one of the vertices of Δ is in all longest cycles of G.*

A 3-tree is defined recursively as follows. A graph that is a triangle is a 3-tree. Any graph obtained from a 3-tree by adding a new vertex and making it adjacent to all the vertices of an existing triangle is also a 3-tree. We say that a graph that is a triangle is a trivial 3-tree, while all the other 3-trees are nontrivial. A graph is a partial 3-tree if it is a subgraph of a 3-tree. Forbidden minors are known for partial 3-trees.

Lemma 3. *[Arnborg et al. 1990] There is no K_5-minor in a partial 3-tree.*

3. 3-trees

Our goal is to show that all longest cycles intersect in 3-trees. Next we will generalize Lemmas 7 and 8 of de Rezende *et al.* [De Rezende et al. 2013] for 3-trees.

Lemma 4. *If K is a K_4 in a graph G, then either G contains a K_5-minor or*

$$\cap\{\mathrm{Comp}_\Delta(v_\Delta) : \Delta \textit{ is a triangle in } K\} = \emptyset,$$

where v_Δ is the vertex of K not in Δ.

Let K be a K_4 in a connected graph G. We say that K is 4-fencing if, for each triangle Δ in K, there is a longest cycle C in G such that C is fenced by Δ, intersects Δ at most twice, and satisfies $\mathrm{Comp}_\Delta(C) = \mathrm{Comp}_\Delta(v_\Delta)$, where v_Δ is the vertex of K not in Δ.

Lemma 5. *If, for every triangle Δ in a nontrivial 3-tree G, there is a longest cycle fenced by Δ and intersecting Δ at most twice, then G contains a 4-fencing K_4.*

The next lemma is basically a corollary of Lemma 2.

Lemma 6. *Let G be a 2-connected graph. If K is a 4-fencing K_4 in G and C is a longest cycle such that $\mathrm{Comp}_K(C) \cap \mathrm{Comp}_\Delta(v_\Delta) = \emptyset$, where Δ is a triangle in K and v_Δ is the vertex in K not in Δ, then there is a vertex in K intersecting all longest cycles in G.*

Corollary 7. *Let G be a 3-tree. Either there exists a vertex intersecting all longest cycles in G or every 4-fencing K_4 in G fences no longest cycle.*

The corollary above implies that, in a 3-tree, either there is a vertex intersecting all longest cycles or every 4-fencing K_4 intersects at least twice every longest cycle, because each such cycle crosses such K_4. Next lemma strengthens this result.

Lemma 8. *Let G be a 3-tree. Either there exists a vertex intersecting all longest cycles in G, or no longest cycle in G 2-touches a 4-fencing K_4 in G.*

So, in a 3-tree G, either there exists a vertex intersecting all longest cycles in G, or every longest cycle intersects at least three times each 4-fencing K_4. If G has a 4-fencing K_4 and no vertex intersects all longest cycles in G, then, for every triangle Δ in this K_4, there must be a longest cycle 3-touching the K_4 at the vertices of Δ.

Theorem 9. *In every 3-tree G, there exists a vertex intersecting all longest cycles in G, that is,* $\mathrm{lct}(G) = 1$.

4. Partial 3-trees

Before proving our main result, we give some basic definitions and notation.

A tree-decomposition [Diestel 2010, p. 337] of a graph G is a pair $(T, \mathcal{V})$ consisting of a tree T and a collection $\mathcal{V} = \{V_t : t \in V(T)\}$ of bags $V_t \subseteq V(G)$, that satisfies the following three conditions:

(T1) $\bigcup_{t \in V(T)} V_t = V(G)$;
(T2) for every $uv \in E(G)$, there exists a bag V_t such that $u, v \in V_t$;
(T3) if a vertex v is in two different bags V_{t_1} and V_{t_2}, then v is also in every bag V_t such that t is on the (unique) path from t_1 to t_2 in T.

The width of $(T, \mathcal{V})$ is the number $\max\{|V_t| - 1 : t \in V(T)\}$, and the tree-width $\text{tw}(G)$ of G is the minimum width of any tree-decomposition of G.

It is known that partial 3-trees are exactly the graphs with tree-width three [Brandstädt et al. 1999]. Sometimes we are interested in particular tree-decompositions. The following proposition is a lemma of Bodlaender.

Proposition 10. *[Bodlaender 1998] If k is the tree-width of a graph G, then G has a tree-decomposition $(T, \mathcal{V})$ of width k such that $|V_t| = k + 1$ for every $t \in V(T)$, and $|V_t \cap V_{t'}| = k$ for every $tt' \in E(T)$.*

Given a node t of T, we say that the connected components of $T - t$ are the branches of T at t [Heinz 2013]. For a node $t' \in V(T - t)$, we denote by $\text{Branch}_t(t')$ the branch of T at t where t' lies. Similarly, for a vertex $v \in V(G)$, we denote by $\text{Branch}_t(v)$ the branch $\text{Branch}_t(q)$ of T at t where q is a node of T such that $v \in V_q$.

Lemma 11. *Let G be a 2-connected partial 3-tree. Let $(T, \mathcal{V})$ be a tree-decomposition of G as described in Proposition 10, and let t be a node of T. If $\text{lct}(G) > 2$, then there exists a longest cycle in G that touches V_t at most once.*

Theorem 12. *For every 2-connected partial 3-tree G, $\text{lct}(G) \leq 2$.*

Proof. Let $(T, \mathcal{V})$ be a tree-decomposition of G as described in Proposition 10. Assume by contradiction that $\text{lct}(G) > 2$. By Lemma 11, for every node $t \in V(T)$, there exists a longest cycle that touches V_t at most once. We create an auxiliary digraph D, that admits anti parallel arcs, as follows. The nodes of D are exactly the nodes of T. Given a node $t \in V(T)$, we add the arc tt' in D if there is a longest cycle C fenced by V_t such that $\text{Branch}_t(t') = \text{Branch}_t(C)$. Then, every node of D is the tail of some arc in D.

Let tt' be the last arc of a maximal directed path in D. As T is a tree, $t't$ is also an arc, which implies that there exist two longest cycles C_t and $C_{t'}$ such that $\text{Branch}_t(C_t) = \text{Branch}_t(t')$ and $\text{Branch}_{t'}(C_{t'}) = \text{Branch}_{t'}(t)$, where C_t touches V_t at most once, and $C_{t'}$ touches $V_{t'}$ at most once. Note that the bags where the vertices of C_t lie are only in $\text{Branch}_t(t') \cup \{t\}$, and that the bags where the vertices of $C_{t'}$ lie are only in $\text{Branch}_{t'}(t) \cup \{t'\}$. As $\text{Branch}_t(t')$ and $\text{Branch}_{t'}(t)$ are disjoint, $C_t \cap C_{t'} \subseteq V_t \cup V_{t'}$. Note that C_t does not contain the only vertex in $V_t \setminus V_{t'}$, and that $C_{t'}$ does not contain the only vertex in $V_{t'} \setminus V_t$. But then C_t and $C_{t'}$ intersect each other in just one vertex, a contradiction to the fact that G is 2-connected. $\square$

5. Final remarks

Similar proofs can lead to more general results. Namely, that $\mathrm{lct}(G) \leq \omega(G) - 3$ for every 2-connected chordal graph, where $\omega(G)$ is the size of a maximum clique of G, and that $\mathrm{lct}(G) \leq k - 1$ for every 2-connected graph of tree-width k. This would imply that $\mathrm{lct}(G) = 1$ for 2-connected partial 2-trees and for 2-connected planar graphs that are also chordal.

Similar but weaker results can be obtained when considering paths instead of cycles. One difficulty that arises is that paths can have their ends in different connected components. That is, given a longest path P with ends x and y, and a triangle Δ, we can have that $\mathrm{Comp}_\Delta(x) \neq \mathrm{Comp}_\Delta(y)$. This generates more possibilities, and for this reason the results are not as strong as for cycles.

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