The 1,2,3-Conjecture for powers of paths and powers of cycles

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Abstract. *A labelling of a graph G is a mapping $\pi\colon S \to \mathcal{L}$, where $\mathcal{L} \subset \mathbb{R}$ and $S = E(G)$ or $S = V(G) \cup E(G)$. If $S = E(G)$, π is an $\mathcal{L}$*-edge-labelling *and, if $S = V(G) \cup E(G)$, π is an $\mathcal{L}$*-total-labelling. *For each $v \in V(G)$, the colour of v under π is defined as $c_\pi(v) = \sum_{uv\in E(G)} \pi(uv)$ if π is an $\mathcal{L}$-edge-labelling; and $c_\pi(v) = \pi(v) + \sum_{uv\in E(G)} \pi(uv)$ if π is an $\mathcal{L}$-total-labelling. The pair (π, c_π) is a neighbour-distinguishing-$\mathcal{L}$-edge (total)-labelling if $\pi\colon S \to \mathcal{L}$ is an edge (total)-labelling and $c_\pi(u) \neq c_\pi(v)$, for every edge $uv \in E(G)$. The 1,2,3-Conjecture states that every simple graph with no isolated edge has a neighbour-distinguishing-$\mathcal{L}$-edge-labelling with $\mathcal{L} = \{1, 2, 3\}$. In this work, we verify the 1,2,3-Conjecture for powers of paths and powers of cycles and we also show that powers of cycles have a neighbour-distinguishing-$\{a, b\}$-total-labelling, $a, b \in \mathbb{R}$, $a \neq b$.*

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. We denote an edge $e \in E(G)$ by uv where u and v are its endpoints. An *element* of G is a vertex or an edge of G. As usual, the degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$. We say that G is *k-regular* if all of its vertices have degree k.

For $S = E(G)$ or $S = V(G) \cup E(G)$ and a set $\mathcal{L} \subset \mathbb{R}$, a *labelling* of G is a mapping $\pi\colon S \to \mathcal{L}$. If $S = E(G)$, π is an *$\mathcal{L}$-edge-labelling* and, if $S = V(G) \cup E(G)$, π is an *$\mathcal{L}$-total-labelling*. Given a labelling π of G, we define $c_\pi\colon V(G) \to \mathcal{C}$, $\mathcal{C}$ a set of *colours*, such that, for each $v \in V(G)$, $c_\pi(v) = \sum_{uv\in E(G)} \pi(uv)$, if π is an $\mathcal{L}$-edge-labelling, and $c_\pi(v) = \pi(v) + \sum_{uv\in E(G)} \pi(uv)$, if π is an $\mathcal{L}$-total-labelling. The mapping c_π is a *proper-vertex-colouring* of G if $c_\pi(u) \neq c_\pi(v)$, for every edge $uv \in E(G)$. We say that the pair (π, c_π) is a *neighbour-distinguishing-$\mathcal{L}$-edge-labelling* when π is an $\mathcal{L}$-edge-labelling and c_π is a proper-vertex-colouring. Similarly, taking π an $\mathcal{L}$-total-labelling in the previous definition, we say that (π, c_π) is a *neighbour-distinguishing-$\mathcal{L}$-total-labelling*. In this work, $[k]$ denotes the set of consecutive integers $\{1, \ldots, k\}$.

In 2004, Karónski et al. [4] proposed the problem of determining the least positive integer k needed to obtain a neighbour-distinguishing-$[k]$-edge-labelling of an arbitrary simple graph G without isolated edges. This problem has attracted attention and the least value of k known is due to Kalkowski et al. [3], who proved that every graph with no isolated edge has a neighbour-distinguishing-$[5]$-edge-labelling. In their work, Karónski et al. observed that all the families of graphs they studied have a neighbour-distinguishing-$[3]$-edge-labelling. Based on this, they posed the following conjecture.

Conjecture 1 (Karónski et al. [4]). *If G is a simple graph with no isolated edge, then G has a neighbour-distinguishing-$[3]$-edge-labelling.*

Conjecture 1 is known as the *1,2,3-Conjecture*. Karónski et al. also observed that almost all graphs they investigated had a neighbour-distinguishing-$[2]$-edge-labelling. Later, Dudek and Wajc [1] proved that deciding whether a graph has a neighbour-distinguishing-$[2]$-edge-labelling is $\mathcal{NP}$-complete. Since then, neighbour-distinguishing-$[2]$-edge-labellings have been investigated for families of graphs. Recently, Thomassen et al. [9] completely characterized the bipartite graphs that have a neighbour-distinguishing-$[2]$-edge-labelling and, using results from nowhere-zero-3-flows [8], proved that every nonbipartite $(6p-7)$-edge-connected graph of chromatic number at most p (where p is any odd natural number ≥ 3) has a neighbour-distinguishing-$[2]$-edge-labelling.

Motivated by the neighbour-distinguishing-edge-labelling problem, Przybyło and Woźniak [7] introduced the related problem of neighbour-distinguishing-total-labellings and studied it for some classes of graphs. They also posed the *1,2-Conjecture* that states that every simple graph has a neighbour-distinguishing-$[2]$-total-labelling. Hulgan et al. [2] generalised this problem considering $\mathcal{L} = \{a, b\}$, $a, b \in \mathbb{R}$, $a \neq b$, approaching it for families of graphs.

In this work, we verify the 1,2,3-Conjecture for powers of cycles and powers of paths. Moreover, we show that every power of cycles has a neighbour-distinguishing-$\{a, b\}$-total-labelling, for $a, b \in \mathbb{R}$, $a \neq b$.

2. Preliminaries

Given a simple graph G and two distinct vertices $u, v \in V(G)$, the *distance* between u and v in G is the number of edges in a shortest path connecting u and v, and is denoted by $d_G(u, v)$. The *k-th power* G^k of a graph G is the simple graph that has vertex set $V(G^k) = V(G)$, with distinct vertices u, v being adjacent in G^k if and only if $d_G(u, v) \leq k$.

As usual, a path with n vertices is denoted by P_n and a cycle with n vertices is denoted by C_n. When $G \cong P_n$, graph G^k is called *power of paths* and is denoted by P_n^k. Similarly, when $G \cong C_n$, graph G^k is called *power of cycles* and is denoted by C_n^k. Note that $P_n^k \cong K_n$, when $k \geq n-1$, and $C_n^k \cong K_n$, when $k \geq \lfloor n/2 \rfloor$. Since the 1,2,3-Conjecture has been verified for paths, cycles and complete graphs [4], in this work we consider only powers of paths P_n^k with $1 < k < n-1$ and powers of cycles C_n^k with $1 < k < \lfloor n/2 \rfloor$. The next results, related to neighbour-distinguishing labellings of graphs, are used in the proofs of Section 3.

Proposition 2. *Let (π, c_π) be a neighbour-distinguishing-$[k]$-total-labelling of a graph G. If $\pi(v) = p$, for every $v \in V(G)$, then $(\pi', c_{\pi'})$ such that $\pi'(uv) = \pi(uv)$, for every $uv \in E(G)$, is a neighbour-distinguishing-$[k]$-edge-labelling.* □

Lemma 3 (Przybyło and Woźniak [7])**.** *If $G \cong K_n$, then G has a neighbour-distinguishing-$[2]$-total-labelling (π, c_π), called* canonical labelling, *such that, either $\mathcal{C} = \{n, \ldots, 2n-1\}$ or $\mathcal{C} = \{n+1, \ldots, 2n\}$. Moreover, if either $n = 3$ and $\mathcal{C} = \{4, 5, 6\}$ or $n \geq 4$, then π has at least two vertices with label 2.* □

Lemma 4. *Let $G \cong K_n$ with $n \geq 3$. Let $(\pi', c_{\pi'})$ be a canonical labelling, with $\mathcal{C} = \{4, 5, 6\}$ for $n = 3$. Then, $(\pi', c_{\pi'})$ can be modified so as to obtain a neighbour-distinguishing-$[3]$-total-labelling (π, c_π) such that: (i) for every $v \in V(G)$, $\pi(v) = 1$; and (ii) if $c_{\pi'}(v) = \max_{u \in V(G)}\{c_{\pi'}(u)\}$, then $c_\pi(v) \in \{c_{\pi'}(v), c_{\pi'}(v) + 1\}$; otherwise, $c_\pi(v) = c_{\pi'}(v)$.*

Proof. Let G and $(\pi', c_{\pi'})$ as stated in the hypothesis. Let $S = \{v \in V(G) \colon \pi'(v) = 2\}$. Let M be a maximum matching of $G[S]$. Adjust notation so that, if $|S| \equiv 1 \pmod 2$, the unsaturated vertex $u \in S$ has $c_{\pi'}(u) = \min_{v \in S}\{c_{\pi'}(v)\}$. First, define $\pi(v) = 1$, if $v \in S$, and $\pi(v) = \pi'(v)$, otherwise. If $e \in M$, $\pi(e) = \pi'(e)+1$, otherwise $\pi(e) = \pi'(e)$. Note that every vertex $v \in S$ has its label decreased by one and, if $|S| \equiv 0 \pmod 2$, v has the label of exactly one of its incident edges increased by one. Therefore, $c_\pi(v) = c_{\pi'}(v)$. If $|S| \equiv 1 \pmod 2$, let $w \in S$ such that $c_{\pi'}(w) = \max_{v \in V(G)}\{c_{\pi'}(v)\}$. Define $\pi(uw) = \pi'(uw) + 1$. This implies that $c_\pi(u) = c_{\pi'}(u)$ and $c_\pi(w) = c_{\pi'}(w) + 1$. Since $c_{\pi'}(w) = \max_{v \in V(G)}\{c_{\pi'}(v)\}$, we conclude that c_π is a proper-vertex-colouring of G. Moreover, $\pi \colon V(G) \cup E(G) \to \{1, 2, 3\}$ and the result follows. □

Theorem 5 (Luiz et al. [5, 6]). *If $G \cong P_n^k$ or $G \cong C_n^k$, then G has a neighbour-distinguishing-$[2]$-total-labelling.* □

3. Results

In this section, we state our main results.

Theorem 6. *If G is a simple graph such that $G \cong P_n^k$ or $G \cong C_n^k$, then G has a neighbour-distinguishing-$[3]$-edge-labelling.*

Outline of the proof. Let $G \cong P_n^k$ or $G \cong C_n^k$. We assume G is not a path, a cycle, or a complete graph, since the result is known for these cases. If $G \cong P_n^k$, with $n \geq 2k + 2$, or $G \cong C_n^k$, we write $n = \alpha(k + 1) + r$ such that $0 \leq r \leq k$, $\alpha \geq 2$, and $r, \alpha \in \mathbb{N}$. In these cases, we take a partition $\mathcal{P} = \{B^0, \ldots, B^\alpha\}$ of $V(G)$ into *blocks* B^i such that each block induces a complete graph. Note that $\mathcal{P}$ comprises α blocks with $k + 1$ vertices and one block with r vertices. Now, if $G \cong P_n^k$ with $k + 1 < n < 2k + 2$, we take a partition $\mathcal{P} = \{B^0, B^1, B^2\}$ of $V(G)$ such that $|B^0| = |B^2| = \lfloor (n - k)/2 \rfloor$ and $k \leq |B^1| \leq k+1$.

We consider seven cases: (i) $G \cong P_n^k$ and $n \geq 2k + 2$; (ii) $G \cong P_n^k$ and $k + 1 < n < 2k + 2$; (iii) $G \cong C_n^k$, $k \geq 3$ and $r \neq 1$; (iv) $G \cong C_n^k$, $k \geq 3$ and $r = 1$; (v) $G \cong C_n^k$, $k = 2$ and $r = 0$; (vi) $G \cong C_n^k$, $k = 2$ and $r = 1$; and (vii) $G \cong C_n^k$, $k = r = 2$.

In order to prove the result, by Proposition 2, it suffices to show that G has a neighbour-distinguishing-[3]-total-labelling (π, c_π) such that $\pi(v) = 1$ for every $v \in V(G)$. Such a labelling (π, c_π) is obtained by modifying the neighbour-distinguishing-[2]-total-labelling $(\pi', c_{\pi'})$ of G constructed in the proof of Theorem 5, using the technique presented in the proof of Lemma 4: for each vertex $v \in V(G)$, with $\pi'(v) = 2$, we assign $\pi(v) = 1$ and increase the labels of some selected edges by one, maintaining the property that any two adjacent vertices have distinct colours.

In this extended abstract, we exemplify this construction for case (i), for which $G \cong P_n^k$ and $n \geq 2k + 2$. Let $(v_0, \ldots, v_{n-1})$ be a linear order of $V(P_n)$. Thus, for $0 \leq i \leq \alpha - 2$, block B^i comprises the $k + 1$ consecutive vertices, starting from $v_{i(k+1)}$ and following the linear order. Block $B^{\alpha-1}$ comprises the set of r consecutive vertices starting from $v_{(\alpha-1)(k+1)}$, and the remaining $k + 1$ vertices comprise block B^α. Note that, if $r = 0$, then $B^{\alpha-1} = \emptyset$. Considering this partition $\mathcal{P}$ of $V(G)$, $(\pi', c_{\pi'})$ is defined as follows: (i) every element in $G[B^0]$ receives label 2; (ii) every element in $G[B^\alpha]$ receives label 1; (iii) for $1 \leq i \leq \alpha - 1$, $G[B^i]$ receives a canonical labelling with colour set $\mathcal{C} = \{|B^i| + 1, \ldots, 2|B^i|\}$, such that any two vertices $v_k, v_l \in B^i$ have $c_{\pi'}(v_k) < c_{\pi'}(v_l)$ if and only if $k < l$; and (iv) the remaining edges receive label 1.

By the definition of $(\pi', c_{\pi'})$, block B^α has no vertices with label 2. Moreover, by Lemma 3, for $0 \leq i \leq \alpha - 2$, the complete subgraph $G[B^i]$ has at least two vertices with label 2. Thus, we modify $(\pi', c_{\pi'})$ so as to obtain (π, c_π) as follows. For $0 \leq i \leq \alpha-2$, we apply Lemma 4 to $G[B^i]$ obtaining: (i) for every $v \in V(B^i)$, $\pi(v) = 1$; and (ii) if $c_{\pi'}(v) = \max_{v \in B^i}\{c_{\pi'}(v)\}$, then $c_\pi(v) \in \{c_{\pi'}(v), c_{\pi'}(v) + 1\}$; otherwise, $c_\pi(v) = c_{\pi'}(v)$. In order to conclude (π, c_π), it remains to analise block $B^{\alpha-1}$. If the number of vertices with label 2 in $B^{\alpha-1}$ is even, apply Lemma 4 to $G[B^{\alpha-1}]$. Otherwise, (i) choose any $w \in B^{\alpha-1}$ with $\pi'(w) = 2$ and take $x \in B^{\alpha-2}$ with maximum colour under $c_{\pi'}$; (ii) apply Lemma 4 to $G[B^{\alpha-1} \backslash \{w\}]$; (iii) and let $\pi(w) = 1$ and $\pi(wx) = \pi'(wx) + 1$. Note that the resulting labelling is a neighbour-distinguishing-[3]-total-labelling (π, c_π) of G. In order to see this, note that $\pi(e) \in \{1, 2, 3\}$, $e \in E(G)$, $c_\pi(v) = c_{\pi'}(v)$, $v \in V(G)\backslash\{v_{i(k+1)+k} \colon 0 \leq i \leq \alpha - 2\}$, and $c_{\pi'}(v) \leq c_\pi(v) \leq c_{\pi'}(v) + 2$, otherwise. Note that the last mentioned vertices, that eventually had their colours changed, are pairwise nonadjacent and have colours in π' that are greater than the colours of their neighbours. Therefore, any two adjacent vertices have distinct colours under c_π, and the result follows. □

Hulgan et al. [2] proved that, for $a, a', b, b' \in \mathbb{R}$, with $a \neq b$ and $a' \neq b'$, a regular graph G has a neighbour-distinguishing-$\{a, b\}$-total-labelling if and only if it has a neighbour-distinguishing-$\{a', b'\}$-total-labelling. By noting that every power of cycles is a regular graph and has a neighbour-distinguishing-[2]-total-labelling, we obtain the following corollary.

Corollary 7. *Every power of cycles has a neighbour-distinguishing-$\{a, b\}$-total-labelling, for $a, b \in \mathbb{R}$, $a \neq b$.* □

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