

Facility Leasing with Penalties

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***Abstract.** In this paper we study the facility leasing problem with penalties. We present a 3-approximation primal-dual algorithm, based on the algorithms by Nagarajan and Williamson for the facility leasing problem and by Charikar, Khuller, Mount and Narasimhan for the facility location problem with penalties.*

1. Introduction

In the **facility location problem**, it is given a metric space (V, d) , a set $F \subseteq V$ of facilities, an opening cost for each facility, and a set $D \subseteq V$ of clients. The goal is to choose a subset of facilities to open and an assignment between clients and facilities which minimize the cost of opening the facilities plus the sum of the distances from each client to its corresponding facility. This problem does not admit a polynomial-time algorithm with approximation factor smaller than 1.463 unless $P = NP$ [Sviridenko 2002]. Currently the best approximation factor is 1.488 [Li 2013].

In the **facility location problem with penalties**, we may leave a client j unassigned if we choose to pay a penalty π_j . I.e., we must select a subset of the facilities to open, and a subset of the clients to assign to open facilities. The cost of a solution is the cost of opening facilities, plus the sum of the distances from each assigned client to its corresponding facility, plus the penalty cost for unassigned clients. The facility location problem reduces to this problem if we set $\pi_j = \infty$ for every client j . Currently, a 1.5148-approximation algorithm is known for this problem [Li et al. 2015]; there is a simpler 3-approximation algorithm [Charikar et al. 2001].

In the **facility leasing problem**, clients are distributed along the time, and instead of opening facilities permanently, we may lease each facility for one of K different durations $\delta_1, \dots, \delta_K$. The cost for leasing a facility p for δ_k units of time is $\gamma(p, k) \in \mathbb{R}_+$; it depends on the facility position, as in the traditional facility location problem, but also on the leasing type k . We may assume that leasing costs respect economies of scale, i.e., it is more cost-effective to lease facilities for longer periods. We wish to select a set of facility leases that serve the clients and minimizes the leasing costs plus the sum of the distances from each client to the facility lease that serves it. This problem has a 3-approximation primal-dual algorithm [Nagarajan and Williamson 2013].

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Our Contribution. We study the combination of the two previous problems, which we call the **facility leasing problem with penalties** (FLEP): facilities are leased instead of permanently opened, and some clients may be left unassigned by paying a penalty cost. We obtain a 3-approximation algorithm by combining algorithms for the facility leasing problem and for the facility location problem with penalties.

Related Work. For every *covering problem* that admits an α -approximation, the corresponding problem with submodular penalties admits a $(1 - e^{-1/\alpha})^{-1}$ -approximation [Li et al. 2015]. Combining this approach with the algorithm by Nagarajan and Williamson ($\alpha = 3$), one obtains a 3.5277-approximation algorithm for the facility leasing problem with submodular penalties. Note that FLEP is a particular case of this problem, but our algorithm has smaller approximation ratio.

Text Organization. In Section 2 we define some notation and we present a formal definition of FLEP, as well as its primal and dual formulations. In Section 3 we describe our algorithm, and in Section 4 we present some conclusions and future research directions.

2. Notation and Problem Definition

Let $[K] := \{1, \dots, K\}$ be the set of lease types. We denote a facility lease by a triple $f = (p_f, k_f, t_f)$, in which $p_f \in V$ is the point where f is located, $k_f \in [K]$ is a leasing type, and $t_f \in \mathbb{Z}_+$ is the instant in which the lease for f begins. Let $\mathcal{F} := F \times [K] \times \mathbb{Z}_+$. We denote a client by a triple $j = (p_j, \pi_j, t_j)$, in which $p_j \in V$ is the point where j is located, $\pi_j \in \mathbb{R}_+$ is the penalty for not assigning a facility lease to j , and t_j is the instant in which j arrives. For simplicity, we write δ_f instead of $\delta(k_f)$, and γ_f instead of $\gamma(p_f, k_f)$, for $f = (p_f, k_f, t_f) \in \mathcal{F}$. Also, for $f = (p_f, k_f, t_f) \in \mathcal{F}$ and $j = (p_j, \pi_j, t_j) \in V \times \mathbb{R}_+ \times \mathbb{Z}_+$, we define the distance from j to f as $d(j, f) := d(p_j, p_f)$ if $t_j \in [t_f, t_f + \delta_f)$, and $d(j, f) := \infty$ otherwise; i.e., the distance from j to f is infinite if f does not cover t_j .

Problem FLEP($V, d, F, K, \gamma, \delta, D$): *The input consists of a set of points V , a distance function $d : V \times V \mapsto \mathbb{R}_+$ satisfying symmetry and triangle inequality, a set $F \subseteq V$ of potential facilities, an integer $K > 0$ that represents the number of lease types, a cost $\gamma(p, k) \in \mathbb{R}_+$ for leasing facility $p \in F$ with leasing type $k \in [K]$, a function $\delta : [K] \mapsto \mathbb{N}$ that maps each lease type to a length in days, and a set $D \subseteq V \times \mathbb{R}_+ \times \mathbb{Z}_+$ of clients in the form $j = (p_j, \pi_j, t_j)$. A solution consists of a set $X \subseteq \mathcal{F} := F \times [K] \times \mathbb{Z}_+$ of facility leases in the form $f = (p_f, k_f, t_f)$, and a function $a : D \mapsto X \cup \{\text{null}\}$ that maps each client j to some $f \in X$ such that $t_j \in [t_f, t_f + \delta_f)$ or to null. The goal is to find a solution (X, a) which minimizes $\sum_{f \in X} \gamma_f + \sum_{j \in D: a(j) \neq \text{null}} d(j, a(j)) + \sum_{j \in D: a(j) = \text{null}} \pi_j$.*

The problem has the following primal formulation:

$$\begin{aligned}
& \text{minimize} && \sum_{f \in \mathcal{F}} \gamma_f \cdot y_f + \sum_{j \in D} \sum_{f \in \mathcal{F}} d(j, f) \cdot x_{jf} + \sum_{j \in D} \pi_j \cdot z_j \\
& \text{subject to} && x_{jf} \leq y_f \quad \forall f \in \mathcal{F}, j \in D \\
& && \sum_{\substack{f \in \mathcal{F} \\ t_j \in [t_f, t_f + \delta_f)}} x_{jf} + z_j \geq 1 \quad \forall j \in D \\
& && x_{jf}, y_f, z_j \in \{0, 1\} \quad \forall f \in \mathcal{F}, j \in D
\end{aligned}$$

Variable y_f indicates whether facility f was leased, variable x_{jf} indicates whether client j was served by facility lease f , and variable z_j indicates whether the algorithm decided to

pay the penalty associated with not assigning j . The relaxation of the dual problem is:

$$\begin{aligned}
& \text{maximize} && \sum_{j \in D} \alpha_j \\
& \text{subject to} && \sum_{j \in D} (\alpha_j - d(j, f))_+ \leq \gamma_f \quad \forall f \in \mathcal{F} \\
& && \alpha_j \leq \pi_j \quad \forall j \in D \\
& && \alpha_j \geq 0 \quad \forall j \in D
\end{aligned}$$

The dual has the following economical interpretation: each client j is willing to pay α_j to connect itself to some facility lease. Part of this value covers the distance to the facility; the other part is a contribution to pay the facility leasing cost. However, a client is not willing to pay more than its penalty.

3. Algorithm

Our algorithm is based on the algorithm for the facility leasing problem, and on an algorithm for the facility location problem with penalties [Nagarajan and Williamson 2013, Charikar et al. 2001]. We say that client j **reaches** facility lease f if $\alpha_j \geq d(j, f)$.

Algorithm PRIMAL-DUAL-FLEP($V, d, F, K, \gamma, \delta, D$)

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01   $X \leftarrow \emptyset, S \leftarrow D, \alpha_j \leftarrow 0$  for every  $j \in D$ 
02  while  $S \neq \emptyset$  do
03    increase  $\alpha_j$  uniformly for every  $j \in S$  until
04      (a)  $\alpha_j = d(j, f)$  for some  $j \in S$  and  $f \in X$ 
05      or (b)  $\gamma_f = \sum_{j \in D} (\alpha_j - d(j, f))_+$  for some  $f \in \mathcal{F} \setminus X$ 
06      or (c)  $\alpha_j = \pi_j$  for some  $j \in S$ 
07   $X \leftarrow X \cup \{f \in \mathcal{F} \setminus X : f \text{ satisfies (b)}\}$ 
08   $S \leftarrow S \setminus \{j \in S : \alpha_j \geq \pi_j \text{ or } j \text{ reaches some } f \in X\}$ 
09  build the graph  $G_X$  with
10     $V[G_X] \leftarrow X, E[G_X] \leftarrow \{(f, f') : \exists j \in D : j \text{ reaches both } f \text{ and } f'\}$ 
11  build a maximal independent set  $X'$  in  $G_X$  greedily in non-increasing order of  $\delta$ 
12   $\hat{X} \leftarrow \{(p_f, k_f, t_f - \delta_k), f, (p_f, k_f, t_f + \delta_k) : f \in X'\}$ 
13  for every  $j \in D$  do
14    if  $j$  reaches some  $f \in X$ 
15      then  $a(j) \leftarrow \arg \min_{f' \in \hat{X}} \{d(j, f')\}$ 
16      else  $a(j) \leftarrow \text{null}$ 
17  return  $(\hat{X}, a)$ 

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The algorithm maintains a dual variable α_j for each client j , a set X of temporarily leased facilities, and a set S of the clients whose dual variable still is being increased, which initially is the whole set of clients D . Increasing pauses when: (a) a client reaches an already temporarily leased facility, or (b) the sum of the contributions towards a facility lease pays for its cost, or (c) the dual variable reaches the penalty cost for some client. We then add to X the facilities that reach condition (b), and remove from S the clients that reach some temporarily leased facility or whose dual variable pays for the penalty cost, and then proceed increasing the remaining dual variables until S becomes empty.

After this phase, we build an interference graph G_X between the facility leases in X . Graph G_X has vertex set X and has an edge between facilities f and f' if there is some client that reaches both f and f' . Then, we sort set X in non-increasing order of lease duration and build a maximal independent set X' in a greedy manner; i.e., we visit set X in that order and add a facility f to X' if there is no other facility lease $f' \in X'$ reached by some client that reaches f . Thus X' satisfies the following properties:

1. Every client reaches at most one facility lease in X' ;
2. If f and f' in X are reached by the same client j , and if $f' \in X'$, then $\delta_f \leq \delta_{f'}$.

Note that there may be some client j that reaches some f in X but is not covered by any facility lease in X' . However, remember that some $f' \in X'$ shares a reaching client j' with f , thus $\delta_f \leq \delta_{f'}$ and the intervals covered by facility leases f and f' overlap. Then, we buy \hat{X} , which has three copies of f' , beginning at instants $t_{f'} - \delta_{f'}$, $t_{f'}$ and $t_{f'} + \delta_{f'}$. Thus, the interval formed by those three facilities, which is $[t_{f'} - \delta_{f'}, t_{f'} + 2\delta_{f'}]$, is a superset of interval $[t_f, t_f + \delta_f]$, and therefore one of the facility leases covers t_j .

Finally, if some client j does not reach any facility lease in X , then its dual variable pays for its penalty and we set $a(j)$ to null.

Note that, although the number of potential facility leases is infinite, the algorithm may be implemented in polynomial time in the input size: it is enough to consider, for every facility point, a lease beginning at each instant in which we have a client.

We omit the proof of the following theorem due to space constraints.¹

Theorem 1: *Algorithm PRIMAL-DUAL-FLEP is a 3-approximation.*

4. Conclusion and Future Work

We give a 3-approximation algorithm for the facility leasing problem with penalties. Future work may include studying leasing variants of other facility location problems.

References

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¹A proof is available at <http://www.ime.usp.br/~mslima/appendix-etc2017.pdf>.