On a joint technique for Hajós’ and Gallai’s Conjectures

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Abstract. A path (resp. cycle) decomposition of a graph G is a set of edge-disjoint paths (resp. cycles) of G that covers the edge-set of G. Gallai (1966) conjectured that every graph on n vertices admits a path decomposition of size at most \(\lfloor \frac{n+1}{2} \rfloor\), and Hajós (1968) conjectured that every Eulerian graph on n vertices admits a cycle decomposition of size at most \(\lfloor \frac{n-1}{2} \rfloor\). In this paper, we verify Gallai’s Conjecture for series-parallel graphs, and for graphs with maximum degree 4. Moreover, we show that the only graphs in these classes that do not admit a path decomposition of size at most \(\lfloor n/2 \rfloor\) are isomorphic to \(K_3\), \(K_5\), or \(K_5 - e\). The technique developed here is further used to present a new proof of a result of Granville and Moisiadis (1987) that states that Eulerian graphs with maximum degree 4 satisfy Hajós’ Conjecture.

Resumo. Uma decomposição de um grafo G em caminhos (resp. circuitos) é um conjunto de caminhos (resp. circuitos) arestas-disjuntos de G que cobre o conjunto de arestas de G. Gallai (1966) conjecturou que todo grafo com n vértices admite uma decomposição em caminhos \(D\) tal que \(|D| \leq \lfloor \frac{n+1}{2} \rfloor\), e Hajós (1968) conjecturou que todo grafo Euleriano com n vértices admite uma decomposição em circuitos \(D\) tal que \(|D| \leq \lfloor \frac{n-1}{2} \rfloor\). Neste trabalho, nós provamos a Conjectura de Gallai para grafos série-paralelos, e para grafos com grau máximo 4. Além disso, nós mostramos que os únicos grafos nessas classes que não admitem uma decomposição \(D\) tal que \(|D| \leq \lfloor n/2 \rfloor\) são isomorfos a \(K_3\), \(K_5\), e \(K_5 - e\). A técnica desenvolvida aqui é também usada para apresentar uma nova prova de um resultado de Grainville e Moisiadis (1987) que diz que grafos Eulerianos com grau máximo 4 satisfazem a Conjectura de Hajós.

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1. Introduction

A decomposition $\mathcal{D}$ of a graph $G$ is a set $\{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of $G$ that cover the edge-set of $G$. We say that $\mathcal{D}$ is a path (resp. cycle) decomposition if $H_i$ is a path (resp. cycle) for $i = 1, \ldots, k$. We say that a path (resp. cycle) decomposition $\mathcal{D}$ of a graph (resp. an Eulerian graph) $G$ is minimum if for any path (resp. cycle) decomposition $\mathcal{D}'$ of $G$ we have $|\mathcal{D}| \leq |\mathcal{D}'|$. The size of a minimum path (resp. cycle) decomposition is called the path (resp. cycle) number of $G$, and is denoted by $pn(G)$ (resp. $cn(G)$). In this paper, we focus in the following conjectures concerning minimum path and cycles decompositions of graphs (see [Bondy 2014, Lovász 1968]).

**Conjecture 1 (Gallai, 1966)** If $G$ is a connected graph with $n$ vertices, then $pn \leq \lfloor \frac{n+1}{2} \rfloor$.

**Conjecture 2 (Hajós, 1968)** If $G$ is an Eulerian graph with $n$ vertices, then $cn \leq \lfloor \frac{n-1}{2} \rfloor$.

Although these conjectures are very similar, the results obtained towards their verification are distinct. In 1968, Lovász proved that a graph with $n$ vertices can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles. A consequence of this result is that if $G$ is a graph with at most one vertex of even degree, then $pn(G) = \lfloor n/2 \rfloor$. Pyber (1996) and Fan (2005) extended this result, but the conjecture is still open. In [Botler and Jiménez 2017], one of the authors verified Conjecture 1 for a family of even regular graphs, and Jiménez and Wakabayashi (2014) verified it for a family of triangle-free graphs.

In another direction, Geng, Fang and Li (2015) verified Conjecture 1 for maximal outerplanar graphs and 2-connected outerplanar graphs, and Favaron and Kouider (1988) verified it for Eulerian graphs with maximum degree 4. While we were writing this paper, we learned that Bonamy and Perrett [Bonamy and Perrett 2016] verified Conjecture 1 for graphs with maximum degree 5.

Conjecture 2, on the other hand, was only verified for graphs with maximum degree 4 [Granville and Moisiadis 1987] and for planar graphs [Seyffarth 1992].

In this paper, we present a technique that showed to be useful to deal with both Gallai’s and Hajós’ Conjectures. Our technique consists of finding, given a graph $G$, a special subgraph $H$, which we call a reducing subgraph of $G$, that have small path or cycle number compared to the number of vertices of $G$ that are isolated in $G - E(H)$. In this paper we focus on series–parallel graphs and graphs with maximum degree 4. We verify Gallai’s and Hajós’ Conjectures for these classes in Section 2 and 3, respectively. Due to space limitations, we present only the sketch of some proofs.

2. Reducing subgraphs and Gallai’s Conjecture

Let $G$ be a graph and let $H$ be a subgraph of $G$. Given a positive integer $r$, we say that $H$ is an $r$-reducing subgraph of $G$ if $G - E(H)$ has at least $2r$ isolated vertices and $pn(H) \leq r$. The following lemma arises naturally.

**Lemma 1** Let $G$ be a graph and $H \subseteq G$ be an $r$-reducing subgraph of $G$. If $pn(G - E(H)) \leq \lfloor n/2 \rfloor - r$, then $pn(G) \leq \lfloor n/2 \rfloor$.

In order to verify Conjecture 1 for graphs with maximum degree 4, we first extend the results in [Geng et al. 2015] by proving that Gallai’s Conjecture holds for series–parallel graphs, which are precisely the graphs with no subdivision of $K_4$. The proof of the next theorem relies on the fact that series–parallel graphs with at least four vertices
contain at least two non-adjacent vertices of degree at most 2. This fact is easy to verify, since series-parallel graphs are also the graphs with treewidth at most 2.

**Theorem 2** Let $G$ be a connected graph on $n$ vertices. If $G$ has no subdivision of $K_4$, then $pn(G) \leq \lceil n/2 \rceil$ or $G$ is isomorphic to $K_3$.

**Sketch of the proof.** For a contradiction, let $G$ be a minimum counter-example for the statement. It is not hard to verify that $G$ has at least five vertices. Thus, let $u, v$ be two non-adjacent vertices of degree at most 2. We can show that $u$ and $v$ have at most one neighbor in common, which implies that there is a path $P$ containing both $u$ and $v$ as internal vertices. Let $H$ be the graph consisting of $P$ together with the components of $G - E(P)$ that isomorphic to $K_3$. We can show that $H$ is an $r$-reducing subgraph and that $pn(G - E(H)) \leq \lceil n/2 \rceil - r$. Therefore, Lemma 1 concludes the proof.

The same technique verifies Conjecture 1 for planar graphs with girth at least 6.

**Theorem 3** If $G$ is a planar graph on $n$ vertices and girth at least 6, then $pn(G) \leq \lceil n/2 \rceil$.

The next theorem verifies Conjecture 1 for graphs with maximum degree 4.

**Theorem 4** If $G$ is a connected graph on $n$ vertices and has maximum degree 4, then $pn(G) \leq \lceil n/2 \rceil$ or $G$ is isomorphic to $K_3$, $K_5$ or to $K_5^-$.

**Sketch of the proof.** For a contradiction, let $G$ be minimum counter-example for the statement. By Theorem 2, we may suppose that $G$ contains a subdivision $H$ of $K_4$. Let $v_1, v_2, v_3, v_4$ be the vertices of $H$ with degree 3, and let $S$ be the set of edges incidents to $v_i$ in $G - E(H)$, for $i = 1, 2, 3, 4$. The rest of the proof depends on the structure of $G$ induced by $S$. We analyze one of the possible cases. Suppose that there are distinct vertices $x, y$ in $V(G)$ such that $S \subseteq \{xv_1, xv_2, yv_3, yv_4\}$. It is not hard to check that $H + S$ can be decomposed into two paths, and $v_1, v_2, v_3, v_4$ are isolated vertices in $G - E(H) - S$. Now, let $H'$ be the graph consisting of $H + S$ together with the components of $G - E(H) - S$ that are isomorphic to $K_3, K_5$, or $K_5 - e$. Again, we can show that $H'$ is an $r$-reducing subgraph and that $pn(G - E(H) - S) \leq \lceil n/2 \rceil - r$. Lemma 1 concludes the proof.

3. Reducing subgraphs and Hajós’ Conjecture

When dealing with Conjecture 2, the same strategy holds: we first verify Conjecture 2 for graphs with no subdivision of $K_4$, and then we show how to extend subdivisions of $K_4$ in order to obtain a (cycle) reducing subgraph. Given a positive integer $r$, we say that an Eulerian subgraph $H$ of an Eulerian graph $G$ is an $r$-cycle reducing subgraph of $G$ if $G - E(H)$ has at least $2r$ isolated vertices and $cn(G) \leq r$. Analogously to Section 2 we obtain the following Lemma.

**Lemma 5** Let $G$ be an Eulerian graph and $H \subseteq G$ be an $r$-cycle reducing subgraph of $G$. If $cn(G - E(H)) \leq \lceil (n - 1)/2 \rceil - r$, then $cn(G) \leq \lceil (n - 1)/2 \rceil$.

The next theorems are the main results of this section.

**Theorem 6** If $G$ is an Eulerian graph with $n$ non-isolated vertices and with no subdivision of $K_4$, then $cn(G) \leq \lceil (n - 1)/2 \rceil$.

**Sketch of the proof.** For a contradiction, let $G$ be minimum counter-example for the statement. Let $u, v$ be vertices of degree at most 2 in $G$. It is not hard to prove that $G$
is 2-connected, hence there is a cycle $C$ in $G$ containing $u$ and $v$. The cycle $C$ is a 1-cycle reducing subgraph of $G$, and by the minimality of $G$, we have $cn(G - E(C)) \leq \lfloor (n - 1)/2 \rfloor - 1$. Therefore, Lemma 5 concludes the proof.

**Theorem 7** If $G$ is an Eulerian graph with $n$ vertices and maximum degree 4, then $cn(G) \leq \lfloor (n - 1)/2 \rfloor$.

**Sketch of the proof.** For a contradiction, let $G$ be minimum counter-example for the statement. By Theorem 6, we may suppose that $G$ contains a subdivision $H$ of $K_4$. Thus, $G - E(H)$ contains four vertices, say $v_1, v_2, v_3, v_4$, with degree 1. We can suppose, without loss of generality, that $G - E(H)$ contains paths $P, Q$ joining $v_1$ to $v_2$ and $v_3$ to $v_4$, respectively. We can prove that the subgraph $H' = H + P + Q$ is an $r$-cycle reducing subgraph of $G$ and that $cn(G - E(H')) \leq [n/2] - r$. Lemma 5 concludes the proof.

4. Concluding remarks

Reducing subgraphs have allowed us to obtain both new results and new proofs for known results. Also, this work provides literature with a technique that can be applied at the same time to both Gallai’s and Hajós’ Conjectures. In a forthcoming work we apply this technique to verify Conjectures 1 and 2 for partial 3-trees.

**Referências**


