On a joint technique for Hajós' and Gallai's Conjectures *

Fábio Botler[1], Maycon Sambinelli[2], Rafael S. Coelho[3], Orlando Lee[2]

[1]Facultad de Ciencias Físicas y Matematicas – Universidad de Chile (UChile)
Santiago – Chile

[2]Instituto de Computação – Universidade Estadual de Campinas (Unicamp)
Campinas, SP – Brasil

[3]Instituto Federal do Norte de Minas Gerais (IFNMG)
Montes Claros, MG – Brasil

fbotler@dii.uchile.cl

{msambinelli,lee}@ic.unicamp.br

rafael.coelho@ifnmg.edu.br

Abstract. *A path (resp. cycle) decomposition of a graph G is a set of edge-disjoint paths (resp. cycles) of G that covers the edge-set of G. Gallai (1966) conjectured that every graph on n vertices admits a path decomposition of size at most $\lfloor (n+1)/2 \rfloor$, and Hajós (1968) conjectured that every Eulerian graph on n vertices admits a cycle decomposition of size at most $\lfloor (n-1)/2 \rfloor$. In this paper, we verify Gallai's Conjecture for series–parallel graphs, and for graphs with maximum degree 4. Moreover, we show that the only graphs in these classes that do not admit a path decomposition of size at most $\lfloor n/2 \rfloor$ are isomorphic to K_3, K_5 or $K_5 - e$. The technique developed here is further used to present a new proof of a result of Granville and Moisiadis (1987) that states that Eulerian graphs with maximum degree 4 satisfy Hajós' Conjecture.*

Resumo. *Uma decomposição de um grafo G em caminhos (resp. circuitos) é um conjunto de caminhos (resp. circuitos) arestas-disjuntos de G que cobre o conjunto de arestas de G. Gallai (1966) conjecturou que todo grafo com n vértices admite uma decomposição em caminhos $\mathcal{D}$ tal que $|\mathcal{D}| \leq \lfloor (n+1)/2 \rfloor$, e Hajós (1968) conjecturou que todo grafo Euleriano com n vértices admite uma decomposição em circuitos $\mathcal{D}$ tal que $|\mathcal{D}| \leq \lfloor (n-1)/2 \rfloor$. Neste trabalho, nós provamos a Conjectura de Gallai para grafos série-paralelos, e para grafos com grau máximo 4. Além disso, nós mostramos que os únicos grafos nessas classes que não admitem uma decomposição $\mathcal{D}$ tal que $|\mathcal{D}| \leq \lfloor n/2 \rfloor$ são isomorfos a K_3, K_5 e $K_5 - e$. A técnica desenvolvida aqui é também usada para apresentar uma nova prova de um resultado de Grainwille e Moisiadis (1987) que diz que grafos Eulerianos com grau máximo 4 satisfazem a Conjectura de Hajós.*

*This research has been partially supported by CNPq Projects (Proc. 477203/2012-4 and 456792/2014-7), Fapesp Project (Proc. 2013/03447-6). F. Botler is partially supported by CAPES (Proc. 1617829), Millenium Nucleus Information and Coordination in Networks (ICM/FIC RC 130003), and FONDECYT (proyecto N^o3170878). M. Sambinelli is suported by CNPq (Proc. 141216/2016-6). O. Lee is suported by CNPq (Proc. 311373/2015-1 and 477692/2012-5).

1. Introduction

A *decomposition* $\mathcal{D}$ of a graph G is a set $\{H_1, \ldots, H_k\}$ of edge-disjoint subgraphs of G that cover the edge-set of G. We say that $\mathcal{D}$ is a *path* (resp. *cycle*) *decomposition* if H_i is a path (resp. cycle) for $i = 1, \ldots, k$. We say that a path (resp. cycle) decomposition $\mathcal{D}$ of a graph (resp. an Eulerian graph) G is *minimum* if for any path (resp. cycle) decomposition $\mathcal{D}'$ of G we have $|\mathcal{D}| \leq |\mathcal{D}'|$. The size of a minimum path (resp. cycle) decomposition is called the *path* (resp. *cycle*) *number* of G, and is denoted by $\mathrm{pn}(G)$ (resp. $\mathrm{cn}(\mathrm{G})$). In this paper, we focus in the following conjectures concerning minimum path and cycles decompositions of graphs (see [Bondy 2014, Lovász 1968]).

Conjecture 1 (Gallai, 1966) *If G is a connected graph with n vertices, then* $\mathrm{pn} \leq \lfloor \frac{n+1}{2} \rfloor$.

Conjecture 2 (Hajós, 1968) *If G is an Eulerian graph with n vertices, then* $\mathrm{cn} \leq \lfloor \frac{n-1}{2} \rfloor$.

Although these conjectures are very similar, the results obtained towards their verification are distinct. In 1968, Lovász proved that a graph with n vertices can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles. A consequence of this result is that if G is a graph with at most one vertex of even degree, then $\mathrm{pn}(G) = \lfloor n/2 \rfloor$. Pyber (1996) and Fan (2005) extended this result, but the conjecture is still open. In [Botler and Jiménez 2017], one of the authors verified Conjecture 1 for a family of even regular graphs, and Jiménez and Wakabayashi (2014) verified it for a family of triangle-free graphs.

In another direction, Geng, Fang and Li (2015) verified Conjecture 1 for maximal outerplanar graphs and 2-connected outerplanar graphs, and Favaron and Kouider (1988) verified it for Eulerian graphs with maximum degree 4. While we were writing this paper, we learned that Bonamy and Perrett [Bonamy and Perrett 2016] verified Conjecture 1 for graphs with maximum degree 5.

Conjecture 2, on the other hand, was only verified for graphs with maximum degree 4 [Granville and Moisiadis 1987] and for planar graphs [Seyffarth 1992].

In this paper, we present a technique that showed to be useful to deal with both Gallai's and Hajós' Conjectures. Our technique consists of finding, given a graph G, a special subgraph H, which we call a *reducing subgraph* of G, that have small path or cycle number compared to the number of vertices of G that are isolated in $G - E(H)$. In this paper we focus on series–parallel graphs and graphs with maximum degree 4. We verify Gallai's and Hajós' Conjectures for these classes in Section 2 and 3, respectively. Due to space limitations, we present only the sketch of some proofs.

2. Reducing subgraphs and Gallai's Conjecture

Let G be a graph and let H be a subgraph of G. Given a positive integer r, we say that H is an *r-reducing subgraph of G* if $G - E(H)$ has at least $2r$ isolated vertices and $\mathrm{pn}(H) \leq r$. The following lemma arises naturally.

Lemma 1 *Let G be a graph and $H \subseteq G$ be an r-reducing subgraph of G. If* $\mathrm{pn}(G - E(H)) \leq \lfloor n/2 \rfloor - r$, *then* $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$.

In order to verify Conjecture 1 for graphs with maximum degree 4, we first extend the results in [Geng et al. 2015] by proving that Gallai's Conjecture holds for series–parallel graphs, which are precisely the graphs with no subdivision of K_4. The proof of the next theorem relies on the fact that series–parallel graphs with at least four vertices contain at least two non-adjacent vertices of degree at most 2. This fact is easy to verify, since series-parallel graphs are also the graphs with treewidth at most 2.

Theorem 2 *Let G be a connected graph on n vertices. If G has no subdivision of K_4, then $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$ or G is isomorphic to K_3.*

Sketch of the proof. For a contradiction, let G be a minimum counter-example for the statement. It is not hard to verify that G has at least five vertices. Thus, let u, v be two non-adjacent vertices of degree at most 2. We can show that u and v have at most one neighbor in common, which implies that there is a path P containing both u and v as internal vertices. Let H be the graph consisting of P together with the components of $G - E(P)$ that isomorphic to K_3. We can show that H is an r-reducing subgraph and that $\mathrm{pn}(G - E(H)) \leq \lfloor n/2 \rfloor - r$. Therefore, Lemma 1 concludes the proof.

The same technique verifies Conjecture 1 for planar graphs with girth at least 6.

Theorem 3 *If G is a planar graph on n vertices and girth at least 6, then $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$.*

The next theorem verifies Conjecture 1 for graphs with maximum degree 4.

Theorem 4 *If G is a connected graph on n vertices and has maximum degree 4, then $\mathrm{pn}(G) \leq \lfloor n/2 \rfloor$ or G is isomorphic to K_3, K_5 or to K_5^-.*

Sketch of the proof. For a contradiction, let G be minimum counter-example for the statement. By Theorem 2, we may suppose that G contains a subdivision H of K_4. Let v_1, v_2, v_3, v_4 be the vertices of H with degree 3, and let S be the set of edges incidents to v_i in $G - E(H)$, for $i = 1, 2, 3, 4$. The rest of the proof depends on the structure of the subgraph of G induced by S. We analyze one of the possible cases. Suppose that there are distinct vertices x, y in $V(G)$ such that $S \subseteq \{xv_1, xv_2, yv_3, yv_4\}$. It is not hard to check that $H + S$ can be decomposed into two paths, and v_1, v_2, v_3, v_4 are isolated vertices in $G - E(H) - S$. Now, let H' be the graph consisting of $H + S$ together with the components of $G - E(H) - S$ that are isomorphic to K_3, K_5, or $K_5 - e$. Again, we can show that H' is an r-reducing subgraph and that $\mathrm{pn}(G - E(H) - S) \leq \lfloor n/2 \rfloor - r$. Lemma 1 concludes the proof.

3. Reducing subgraphs and Hajós' Conjecture

When dealing with Conjecture 2, the same strategy holds: we first verify Conjecture 2 for graphs with no subdivision of K_4, and then we show how to extend subdivisions of K_4 in order to obtain a (cycle) reducing subgraph. Given a positive integer r, we say that an Eulerian subgraph H of an Eulerian graph G is an *r-cycle reducing subgraph* of G if $G - E(H)$ has at least $2r$ isolated vertices and $\mathrm{cn}(H) \leq r$. Analogously to Section 2 we obtain the following Lemma.

Lemma 5 *Let G be an Eulerian graph and $H \subset G$ be an r-cycle reducing subgraph of G. If $\mathrm{cn}(G - E(H)) \leq \lfloor (n-1)/2 \rfloor - r$, then $\mathrm{cn}(G) \leq \lfloor (n-1)/2 \rfloor$.*

The next theorems are the main results of this section.

Theorem 6 *If G is an Eulerian graph with n non-isolated vertices and with no subdivision of K_4, then $\mathrm{cn}(G) \leq \lfloor (n-1)/2 \rfloor$.*

Sketch of the proof. For a contradiction, let G be minimum counter-example for the statement. Let u, v be vertices of degree at most 2 in G. It is not hard to prove that G

is 2-connected, hence there is a cycle C in G containing u and v. The cycle C is a 1-cycle reducing subgraph of G, and by the minimality of G, we have $\text{cn}(G - E(C)) \leq \lfloor (n-1)/2 \rfloor - 1$. Therefore, Lemma 5 concludes the proof.

Theorem 7 *If G is an Eulerian graph with n vertices and maximum degree 4, then* $\text{cn}(G) \leq \lfloor (n-1)/2 \rfloor$.

Sketch of the proof. For a contradiction, let G be minimum counter-example for the statement. By Theorem 6, we may suppose that G contains a subdivision H of K_4. Thus, $G - E(H)$ contains four vertices, say v_1, v_2, v_3, v_4, with degree 1. We can suppose, without loss of generality, that $G - E(H)$ contains paths P, Q joining v_1 to v_2 and v_3 to v_4, respectively. We can prove that the subgraph $H' = H + P + Q$ is an r-cycle reducing subgraph of G and that $\text{cn}(G - E(H')) \leq \lfloor n/2 \rfloor - r$. Lemma 5 concludes the proof.

4. Concluding remarks

Reducing subgraphs have allowed us to obtain both new results and new proofs for known results. Also, this work provides literature with a technique that can be applied at the same time to both Gallai's and Hajós' Conjectures. In a forthcoming work we apply this technique to verify Conjectures 1 and 2 for partial 3-trees.

Referências

Bonamy, M. and Perrett, T. (2016). Gallai's path decomposition conjecture for graphs of small maximum degree. *ArXiv e-prints*.

Bondy, A. (2014). Beautiful conjectures in graph theory. *European J. Combin.*, 37:4–23.

Botler, F. and Jiménez, A. (2017). On path decompositions of $2k$-regular graphs. *Discrete Mathematics*, 340(6):1405 – 1411.

Fan, G. (2005). Path decompositions and Gallai's conjecture. *J. Combin. Theory Ser. B*, 93(2):117–125.

Favaron, O. and Kouider, M. (1988). Path partitions and cycle partitions of Eulerian graphs of maximum degree 4. *Studia Sci. Math. Hungar.*, 23(1-2):237–244.

Geng, X., Fang, M., and Li, D. (2015). Gallai's conjecture for outerplanar graphs. *Journal of Interdisciplinary Mathematics*, 18(5):593–598.

Granville, A. and Moisiadis, A. (1987). On Hajós' conjecture (minimum cycle-partitions of the edge-set of Eulerian graphs). *Congr. Numer.*, 56:183–187. Sixteenth Manitoba conference on numerical mathematics and computing (Winnipeg, Man., 1986).

Jiménez, A. and Wakabayashi, Y. (2014). On path-cycle decompositions of triangle-free graphs. *ArXiv e-prints*.

Lovász, L. (1968). On covering of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 231–236. Academic Press, New York.

Pyber, L. (1996). Covering the edges of a connected graph by paths. *J. Combin. Theory Ser. B*, 66(1):152–159.

Seyffarth, K. (1992). Hajós' conjecture and small cycle double covers of planar graphs. *Discrete Math.*, 101(1-3):291–306. Special volume to mark the centennial of Julius Petersen's "Die Theorie der regulären Graphs", Part II.