Advances in anti-Ramsey theory for random graphs

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Abstract. *Given graphs G and H, we denote the following property by $G \xrightarrow[\text{p}]{\text{rb}} H$: for every proper edge-colouring of G (with an arbitrary number of colours) there is a rainbow copy of H in G, i.e., a copy of H with no two edges of the same colour. It is known that, for every graph H, the threshold function $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ of this property for the binomial random graph $G(n,p)$ is asymptotically at most $n^{-1/m^{(2)}(H)}$, where $m^{(2)}(H)$ denotes the so-called maximum 2-density of H. In this work we discuss this and some recent results in the study of anti-Ramsey properties in random graphs, and we prove that if $H = C_4$ or $H = K_4$ then $p_H^{\text{rb}} < n^{-1/m^{(2)}(H)}$, which is in contrast with the known facts that $p_{C_k}^{\text{rb}} = n^{-1/m^{(2)}(C_k)}$ for $k \geqslant 7$, and $p_{K_\ell}^{\text{rb}} = n^{-1/m^{(2)}(K_\ell)}$ for $k \geqslant 19$.*

Resumo. *Dados grafos G e H, denotamos a seguinte propriedade por $G \xrightarrow[\text{p}]{\text{rb}} H$: para toda coloração própria das arestas de G (com uma quantidade arbitrária de cores) existe uma cópia multicolorida de H em G, i.e., uma cópia de H sem duas arestas da mesma cor. Sabe-se que, para todo grafo H, a função limiar $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ para essa propriedade no grafo aleatório binomial $G(n,p)$ é assintoticamente no máximo $n^{-1/m^{(2)}(H)}$, onde $m^{(2)}(H)$ denota a assim chamada 2-densidade máxima de H. Neste trabalho discutimos esse e alguns resultados recentes no estudo de propriedades anti-Ramsey para grafos aleatórios, e mostramos que se $H = C_4$ ou $H = K_4$ então $p_H^{\text{rb}} < n^{-1/m^{(2)}(H)}$, que está em contraste com os fatos conhecidos de que $p_{C_k}^{\text{rb}} = n^{-1/m^{(2)}(C_k)}$ para $k \geqslant 7$, e $p_{K_\ell}^{\text{rb}} = n^{-1/m^{(2)}(K_\ell)}$ para $k \geqslant 19$.*

1. Introduction

Let r be a positive integer and let G and H be graphs. We denote by $G \to (H)_r$ the property that any colouring of the edges of G with at most r colours contains a monochromatic copy of H in G. In 1995, Rödl and Ruciński determined the threshold for the property $G(n,p) \to (H)_r$ for all graphs H. The *maximum* 2*-density* $m^{(2)}(H)$ of a graph H is denoted by $m^{(2)}(H) = \max\left\{\frac{|E(J)|-1}{|V(J)|-2} \colon J \subset H,\ |V(J)| \geqslant 3\right\}$, where we suppose $|V(H)| \geqslant 3$.

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Theorem 1 (Rödl and Ruciński [Rödl and Ruciński 1993, Rödl and Ruciński 1995])**.** *Let H be a graph containing a cycle. Then, the threshold function $p_H = p_H(n)$ for the property $G(n,p) \to (H)_r$ is given by $p_H(n) = n^{-1/m^{(2)}(H)}$.*

Given a graph H, we are interested in the following 'anti-Ramsey' type properties of the random graph $G = G(n,p)$, denoted by $G \xrightarrow[\text{p}]{\text{rb}} H$: for every *proper* edge-colouring of G, there exists a *rainbow* copy of H in G, i.e., a copy of H with no two edges of the same colour. The term 'anti-Ramsey' is used in different contexts, but we follow the terminology used in [Kohayakawa et al. 2014, Kohayakawa et al. 2017, Nenadov et al. 2017, Rödl and Tuza 1992]. Since the property $G(n,p) \xrightarrow[\text{p}]{\text{rb}} H$ is increasing for every fixed graph H, we know that it admits a threshold function $p_H^{\text{rb}} = p_H^{\text{rb}}(n)$ [Bollobás and Thomason 1987].

The study of anti-Ramsey properties of random graphs was initiated by Rödl and Tuza, who proved in [Rödl and Tuza 1992] that for every ℓ there exists a fairly small p, such that $G(n,p) \xrightarrow[\text{p}]{\text{rb}} C_\ell$ almost surely. In fact, this result answers positively a question posed by Spencer (see [Erdős 1979], p. 29), who asked whether there are graphs of arbitrarily large girth that contain a rainbow cycle in any proper edge-colouring. We obtained the following result, which implies that $p_H^{\text{rb}} \leqslant n^{-1/m^{(2)}(H)}$ for any fixed graph H.

Theorem 2 (Kohayakawa, Konstadinidis and Mota [Kohayakawa et al. 2014])**.** *If H is a fixed graph, then there exists a constant $C > 0$ such that for $p = p(n) \geqslant Cn^{-1/m^{(2)}(H)}$ we asymptotically almost surely have $G(n,p) \xrightarrow[\text{p}]{\text{rb}} H$.*

The proof of Theorem 2 combines ideas from the regularity method for sparse graphs (see, e.g., [Kohayakawa 1997, Kohayakawa and Rödl 2003, Szemerédi 1978]) and a characterization of quasi-random sparse graphs (see, e.g., [Chung and Graham 2008]). This result was the beginning of a systematic study about anti-Ramsey problems in random graphs. In [Kohayakawa et al. 2017] we proved that for an infinite family of graphs F we have $p_F^{\text{rb}} \ll n^{-1/m^{(2)}(F)}$, which is in contrast with Theorem 1. Before state this result precisely we need one more definition: given a graph H with $m^{(2)}(H) < 2$, put $\beta(H, K_3) = \frac{1}{3}\left(1 + \frac{1}{m^{(2)}(H)}\right)$. Theorem 3 below makes the discussion above precise.

Theorem 3. *Suppose $k \geqslant 4$ and let F be the $(k+1)$-vertex graph composed by a k-vertex graph H with $1 < m^{(2)}(H) < 2$ and a vertex outside of H that is adjacent to two adjacent vertices of H. Then, for a suitably large constant D, if $p \geqslant Dn^{-\beta(H,K_3)}$, then $G(n,p) \xrightarrow[\text{p}]{\text{rb}} F$ almost surely.*

We can easily conclude that for graphs F as in the statement of Theorem 3 we have $p_F^{\text{rb}} \ll n^{-1/m^{(2)}(F)}$ since one can check that $1/m^{(2)}(F) = 1/m^{(2)}(K_3) = 1/2 < \beta(H, K_3) < 1/m^{(2)}(H)$. This makes the following question interesting: What are the graphs H for which $p_H^{\text{rb}} = n^{-1/m^{(2)}(H)}$? Recently, some progress in answering this question was made in [Nenadov et al. 2017], which proved the following result.

Theorem 4 (Nenadov, Person, Škorić and Steger [Nenadov et al. 2017])**.** *Let H be a cycle on at least 7 vertices or a complete graph on at least 19 vertices. Then $p_H^{\text{rb}} = n^{-1/m^{(2)}(H)}$.*

The authors of Theorem 4 remarked that their result could hold for all cycles and cliques of size at least 4. We conjecture that Theorem 4 can indeed be extended to cycles and cliques of size at least 5, but not for C_4 and K_4. In fact, we show that if H is C_4 or K_4, then p_H^{rb} is asymptotically smaller than $n^{-1/m^{(2)}(H)}$.

Theorem 5. *We have* $p_{C_4}^{\mathrm{rb}} = n^{-3/4}$ *and* $p_{K_4}^{\mathrm{rb}} = n^{-7/15}$.

In what follows we give a brief outline of the proof of Theorem 5 for cycles C_4. We remark that the proof for K_4 makes use of similar techniques.

2. Brief outline of the proof of Theorem 5 for C_4

First, we consider the *density* $m(H)$ of a graph H, defined as $m(H) = \max\left\{\frac{|E(J)|}{|V(J)|} : J \subset H,\ |V(J)| \geqslant 1\right\}$. We will use of the following result.

Theorem 6 (Bollobás [Bollobás 2001])**.** *Let H be a fixed graph. Then, $p = n^{-1/m(H)}$ is the threshold for the property that G contains a copy of H.*

Note that for proving the upper bounds it is enough to show that $G(n,p)$ a.s. contains a small graph that forces a rainbow copy of the given graphs in any proper edge-colouring. Since the proof for the upper bounds are much simpler than the proof for the lower bounds, we give the full proof of the upper bound in the case of C_4.

Upper bound for $p_{C_4}^{\mathrm{rb}}$.

Consider the complete bipartite graph $K_{2,4}$ with partition classes $\{a,b\}$ and $\{w,x,y,z\}$. We will first show that any proper colouring of the edges of $K_{2,4}$ contains a rainbow copy of C_4 and then we conclude that for $p \gg n^{-3/4}$ a.s. $G(n,p)$ contains a copy of $K_{2,4}$. Suppose by contradiction that there is a proper colouring χ of $E(K_{2,4})$ with no rainbow copy of C_4. W.l.o.g. let $\chi(aw) = \chi(bx) = 1$ and $\chi(ay) = \chi(bz) = 2$. Since the colouring is proper the edges ax and az get different new colours, say, $\chi(ax) = 3$ and $\chi(az) = 4$. Since the C_4 induced by $\{a,x,b,y\}$ is not rainbow, we have $\chi(by) = 3$. But then the C_4 induced by the vertices $\{a,x,b,z\}$ is rainbow, a contradiction. Therefore, any colouring of the edges of $K_{2,4}$ contains a rainbow C_4. By Theorem 6, if $p \gg n^{-3/4}$, then a.s. $G(n,p)$ contains a copy of $K_{2,4}$. Therefore, a.s. any proper colouring of the edges of $G(n,p)$ contains a rainbow copy of C_4, which implies that $p_{C_4}^{\mathrm{rb}} \leqslant n^{-1/m(K_{2,4})} = n^{-3/4}$.

Lower bound for $p_{C_4}^{\mathrm{rb}}$.

Now let us turn our attention to the lower bounds. Let G and H be graphs. We say that a sequence $F = H_1, \ldots, H_\ell$ of H-copies in G is an H-*chain* if for any $2 \leqslant i \leqslant \ell$ we have $E(H_i) \cap (E(H_1), \ldots, E(H_{i-1})) \neq \varnothing$. Note that a copy of H in G that does not intersect edge-wise with any other copy of H is a maximal H-chain composed by only one copy of H. Furthermore, the edge sets of two distinct maximal H-chains are disjoint. Thus, it is easy to see that each H in G belongs to exactly one maximal H-chain.

Let $G = G(n,p)$ and let $p \ll n^{-3/4}$. The idea is to prove that a.s. there exists a proper colouring of G that contains no rainbow C_4. In this proof we will consider C_4-chains that are maximal with respect to the number of C_4's. The first and more important step is to colour some edges in all maximal C_4-chains so that all C_4's in G will be non-rainbow and this partial colouring will be proper. Then, since all C_4's are coloured we can just give a new colour for each one of the remaining uncoloured edges. For the first step, we use Markov's inequality and the union bound to obtain that a.s.

$$G \text{ does not contains any graph } H \text{ with } m(H) \geqslant 4/3 \text{ and } |V(H)| \leqslant 12. \tag{1}$$

Let $F = C_4^1, \dots, C_4^\ell$ be an arbitrary C_4-chain in G with $m(F) \geqslant 4/3$. Let $2 \leqslant i \leqslant \ell$ be the smallest index such that $F' = C_4^1, \dots, C_4^i$ has density $m(F') \geqslant 4/3$. Then, since $F'' = C_4^1, \dots, C_4^{i-1}$ has density $m(F'') < 4/3$, we can explore the structure of $G(n,p)$ to conclude that $|V(F'')| \leqslant 10$, which implies $|V(F')| \leqslant 12$, a contradiction with (1). Therefore, a.s. $G(n,p)$ contains no copy of C_4-chains F with $m(F) \geqslant 4/3$. Thus, we may assume that all C_4-chains F of G have density $m(F) < 4/3$. In this case, it is possible to analyze carefully the structure of such chains, obtaining the desired colouring, which proves the claimed result.

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